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A THEORY FOR HYDROFOILS OF FINITE SPAN

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CONTENTS

LIST OF SYMBOLS

b	-	1/2 span of the hydrofoil
U	-	free stream velocity
h	-	depth of submergence, measured from trailing edge to the mean free water surface
ζ	-	elevation of the disturbed free surface
φ	-	perturbation velocity potential = $\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$
$\vec{C}$	-	total velocity = $U\vec{i} + \vec{q}$, $\vec{q}$ = grad φ
p	-	pressure of the liquid
ρ	-	density of the liquid
g	-	gravitational acceleration constant
S	-	plan area of the hydrofoil
AR	-	aspect ratio of the hydrofoil = $\frac{(2b)^2}{S}$
$\Gamma(y)$	-	distribution of the circulation strength, $\Gamma_o = \Gamma(0)$
κ	-	g/U^2
D	-	total drag (neglecting frictional drag) = $D_i + D_4$
D_i	-	induced drag = $D_1 + D_2 + D_3$
D_4	-	wave drag
L	-	total lift = $L_o + \Delta L$, L_o - aerodynamic value of L
$u(y)$	-	x-component of $\vec{q}$, evaluated at the hydrofoil
$w(y)$	-	downwash at the hydrofoil
$C_\ell(y)$	-	local lift coefficient distribution (along the span)
$C_d(y)$	-	local drag coefficient distribution (along the span)
$c(y)$	-	chord length of the hydrofoil, $c(0) = c_o$
$\epsilon(y)$	-	downwash angle
α_a	-	absolute angle of attack, measured from the free stream to the zero-lift direction
α_e	-	effective angle of attack = $\alpha_a - \epsilon(y)$
a_a	-	slope of the lift coefficient curve against α_a
a_e	-	slope of the lift coefficient curve against α_e
σ	-	Froude number with respect to h, = $\frac{U^2}{gh}$

LIST OF SYMBOLS (Continued)

β	-	Froude number with respect to b, $= \frac{U^2}{g b}$
λ	-	depth - $\frac{1}{2}$ span ratio $= \frac{h}{b} = \frac{\beta}{\sigma}$
$J_n(z)$	-	Bessel function of the first kind of order n
$K_n(z)$	-	modified Bessel function of the second kind of order n
$K(k)$, $E(k)$	-	complete elliptic integrals of the first and second kind of argument k
$B(k)$, $\mathcal{C}(k)$	-	modified complete elliptic integrals
γ	-	Euler's constant, 0.5772 ···
Υ	-	the ratio $\Gamma_o / \Gamma_{o_\infty}$
C_L	-	over-all lift coefficient $= \frac{L}{\frac{1}{2} \rho U^2 S}$
C_D	-	over-all drag coefficient $= \frac{D}{\frac{1}{2} \rho U^2 S}$

INTRODUCTION

The purpose of the present investigation is to study the hydrodynamic properties of a hydrofoil of finite span moving with constant velocity through deep water at a fixed distance beneath the free surface. The span of the hydrofoil is parallel to the plane of the free surface. This problem has recently attracted a great deal of attention from both the theoretical and the practical points of view. However, the idea of using the hydrofoil as an aid for locomotion over a water surface is certainly not new.

As early as 1898 Forlanini in Italy tried to use hydrofoils for the purpose of supporting high-speed boats. When such a boat travels fast enough, the hydrofoil system installed under the hull can lift the hull above the water surface so that the familiar ship bow-waves are replaced by much weaker waves due to the submerged hydrofoil system. Consequently, the resisting force of the water may be considerably reduced, and the speed-power ratio may thereby be greatly increased. Another practical application of hydrofoils was made in 1911 by Guidoni[1] in Italy who replaced the ordinary floats of a seaplane by hydrofoils. The use of hydrofoils as floats provides greatly improved aerodynamic characteristics of the seaplane during flight. However, the lack of knowledge of the fundamental properties and design data of hydrofoils has prevented the use of hydrofoils in modern applications. The recent revival of interest in the theoretical and experimental studies of this problem is aimed at removing these gaps in basic information.

It is known that, for a wing moving in an infinite fluid medium, one may neglect the influence of gravity and consider only the inertia and viscous effects; the Reynolds condition for dynamical similarity then holds. Under this condition, the nondimensional lift and drag coefficients depend only on the Reynolds number and the geometry of the body. However, the situation for a hydrofoil near the water surface is quite different. The hydrofoil differs from the airfoil not only because of the possible occurrence of cavitation but also through the strong effect of the free water surface. For the motion of a hydrofoil at shallow submergence, one must consider the gravity effect because the wave formations on the free water surface will influence decisively its hydrodynamic properties. It follows that, in this case, the nondimensional lift and drag coefficients will be functions not only of the Reynolds number and its geometry, but also of the Froude number and the cavitation number.

The program of this report is as follows: After a brief survey of the available theoretical and experimental information on the characteristics of hydrofoils, the theory for a hydrofoil of finite span will be formulated. The liquid medium is assumed to be incompressible and nonviscous and of infinite depth. The basic concept of the analysis is patterned after the famous Prandtl wing theory of modern aerodynamics in that the hydrofoil of large aspect ratio may be replaced by a lifting line. The lift distribution along the lifting line is the same as the lift distribution, integrated with respect to the chord of the hydrofoil, along the span direction. The induced velocity field of the lifting line is then calculated by proper consideration of lift distribution along the lifting line, free water surface pressure condition and wave formation. The "local velocity" so determined for flow around each local section perpendicular to the span of the hydrofoil can be considered as that of a two-dimensional flow around a hydrofoil without free water surface. The only additional feature of the flow in this sectional plane is the modification of the geometric angle of attack, as defined by the undisturbed flow, to the so-called effective angle of attack on account of the local induced velocity. Thus the local sectional characteristics to be used can be taken as those of a hydrofoil section in two-dimensional flow without free water surface but may involve cavitation. More precisely, the hydrofoil section at any location of the span has the same hydrodynamic characteristics as if it were a section of an infinite span hydrofoil in a fluid region of infinite extent at a geometric angle of attack equal to α_c, together with proper modification of the free stream velocity. Such characteristics may be obtained by theory or by experiment and should be taken at the same Reynolds number and cavitation number. With this separation of the three-dimensional effects and the two-dimensional effects, the effects of Froude number are singled out. Thus a systematic and efficient analysis of the hydrofoil properties can be made.

The lifting line theory used in this problem is developed through various stages to include the formulation of the direct or indirect problems and also the problem of minimum drag. Finally, detailed calculations of the lift and drag coefficients of a specified hydrofoil are carried out for the case of elliptical distribution of circulation. The effects of the free water surface and the wave formation are examined in detail. Needless to say, the neglect of viscous effects will cause the omission of the frictional drag in the total drag calculation. However, this error can be estimated separately. The problem of the unsteady motion of a hydrofoil is briefly discussed in Appendix II.

SURVEY OF PREVIOUS RESULTS

The investigation of the properties of hydrofoils has been so far mostly experimental. Although there exist some approximate theories, they do not appear to show the effects of wave formation on the hydrodynamic characteristics of the hydrofoil, especially when the submergence is shallow. In Guidoni's work[1], it is claimed that some of the advantages of hydrofoils over conventional ships or floats are better lift-drag ratio and less sensitivity to a rough water surface so that their use may result in a substantial decrease in the structural weight of the hull. On the other hand, a few disadvantages are also mentioned; for example, the danger to the hydrofoil caused by drifting wood and seaweed. At high take-off velocities for the application to seaplanes there is also the possibility of poor performance due to profile cavitation. Some early preliminary work on the hydrofoil problem was carried out by Keldysch and Lawrentjew[2], Kotschin[3] and Wladimirow[4] in ZAHI. Their studies are, in a broad sense, only an extension and modification of Lamb's work on the submerged cylinder[5] and Havelock's work on the sphere beneath a water surface[6, 7]. In Keldysch's work, which is a special case of Kotschin's, the problem of the two-dimensional hydrofoil was solved by replacing the wing by a circular cylinder with circulation. Wladimirow solved the problem of hydrofoils of finite span by replacing the hydrofoil by a horseshoe vortex of constant circulation along the span and by assuming that the free water surface remains flat no matter how close the hydrofoil is to the surface. The calculation then reduces to a two-dimensional problem in the Trefftz's plane behind the wing. This approximation is even poorer than the assumption of infinite Froude number. In 1935, a test of a single hydrofoil (the NACA 0.0009 profile) was carried out by Wladimirow and Frolow in the towing tank of ZAHI[4]. Experimentally it was found that both lift and drag coefficients decrease with the depth of immersion, but the rate of decrease of the lift coefficient was faster than that of drag, especially for small depths. The comparison between theory and experiment on this point was not very satisfactory. About the same time, the problem of cavitation on a submerged obstacle of hydrofoil section such as a ship propeller was approached by Ackeret[8], Walchner[9], Lerbs[10], Martyrer[11], Gutsche[12], and Smith[13]. Around 1937, further contributions to the hydrofoil problem were made by Weinig[14], Tietjens[15], and von Schertel[16]. Weinig gave a preliminary yet exhaustive discussion on hydrofoil and planing problems neglecting completely

the effects of gravity and the free surface elevation; the wave resistance for both cases was estimated approximately afterwards. During the last World War, a series of tests on a number of hydrofoils of different foil sections was carried out at NACA by Land, Benson and Ward[17, 18, 19] in the NACA towing tank. Several interesting results were found. At depths greater than 4 chords, the influence of the free surface is negligibly small. In the range of depths between 4 chords and approximately 1/2 chord, lift and drag coefficients decrease, and the cavitation speed increases, with decrease in depth until the hydrofoil approaches and breaks through the surface at which point a sudden decrease in lift occurs. However, the corresponding values of the lift-drag ratio increase to maximum values when the hydrofoils are near the surface then decrease rapidly with further decrease in depth to values for the planing surfaces. Obviously, these observations indicate that there exists an optimum value of the depth for which the hydrofoil operates most efficiently. The effect of an increase of speed is to reduce the minimum drag coefficient and to make it occur at a corresponding higher value of the lift coefficient until cavitation takes place. Any lower surface cavitation reduces the lift; while complete upper surface cavitation prevents further increase of lift. Under such condition, however, a maximum value of lift as high as 1 ton per square foot was recorded. A profile with sharp leading edge seems desirable for reducing cavitation. The hydrofoil section NACA 16-509 was observed to have favorable hydrodynamic properties. The stability problem of a system of hydrofoils was studied by Imlay[20]. An exhaustive experimental investigation of a single hydrofoil with large dihedral was made by Sottorf[21] in the Göttingen towing tank in which eighteen different profiles were studied; his results have been made available only recently. He found that a thin profile of almost circular segment form with pointed nose and an increased convex camber in the nose area is favorable for laminar flow, and consequently also reduces cavitation, gives higher lift and lift-drag ratio, and minimum spray formation at the junction of the tip of the hydrofoil and the free water surface. The main feature of the depth effects on hydrodynamic properties are in good agreement with NACA results. It was also found that a partial cavitation on the upper surface has a favorable effect provided that it covers less than a half chord because a thin cavitation bubble layer reinforces the flow curvature and thereby increases the lift. In addition, the bubble layer lessens the surface friction by acting as a cushion between the solid surface and the high velocity water stream. However, this favorable range of operation is not very stable.

GENERAL FORMULATION OF THE PROBLEM

Consider a hydrofoil of span $2b$, with arbitrary profile and plan form, moving with constant forward velocity U through deep water at fixed depth of immersion h which is measured from its trailing edge to the mean free water surface. If we choose a coordinate system fixed with respect to the hydrofoil, then in this system the flow picture would appear to be stationary with a uniform free stream velocity U approaching the hydrofoil. Let the x-axis be parallel to the direction of the free stream, the z-axis point upward, with $z = 0$ representing the undisturbed free surface and the trailing edge of the hydrofoil lying between $(0, -b, -h)$ and $(0, b, -h)$ as shown in Fig. 1.

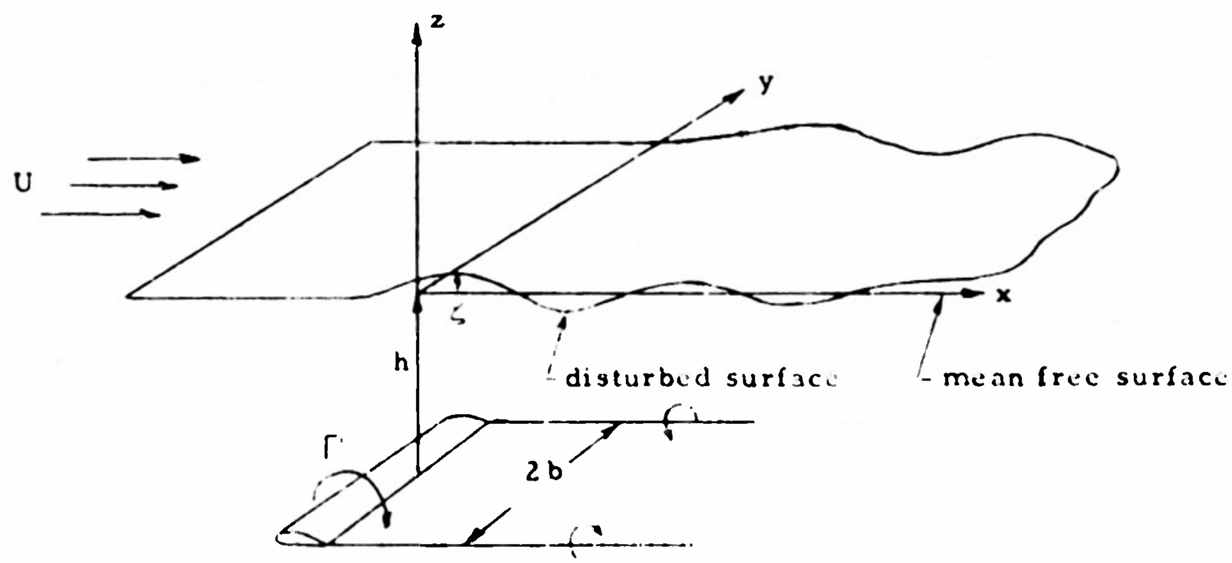

Fig. 1 - Coordinate system for the hydrofoil motion.

The elevation of the disturbed free surface caused by the hydrofoil is denoted by $\zeta(x,y)$ measured from $z = 0$. The liquid medium is assumed to be incompressible and nonviscous so that the condition of irrotationality and continuity implies that the perturbation velocity potential φ satisfies the Laplace equation

$$\nabla^2 \varphi = 0 . \qquad (1)$$

From its solution the total velocity field of the liquid flow can be obtained as

$$\vec{Q} = U\vec{i} + \vec{q}, \qquad \vec{q} \equiv (q_1, q_2, q_3) = \text{grad } \varphi \tag{2}$$

The pressure field may be determined from the Bernoulli integral

$$\frac{p}{\rho} + \frac{1}{2}\vec{Q}\cdot\vec{Q} + gz = \text{const} \tag{3}$$

in which the effect of gravity has been taken into account.

Now let us assume that the resulting motion is such that almost everywhere $|\vec{q}|$ is much less than U so that Eq. (3) may be linearized to yield:

$$\frac{p}{\rho} + U\varphi_x + gz = \text{const.} \tag{4}$$

The corresponding linearized boundary conditions on the disturbed, free surface now become

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = U\frac{\partial\zeta}{\partial x}, \tag{5}$$

and

$$U\left(\frac{\partial\varphi}{\partial x}\right)_{z=0} = -g\zeta. \tag{6}$$

Eq. (5) represents the linearized kinematic condition on the free surface, while Eq. (6) is identical with the dynamic condition $p = p_o$ = constant on the disturbed surface. It may be noted that gravity affects the kinematics through the boundary conditions (5) and (6) and thereafter influences the dynamics through Eq. (4), even though it does not appear explicitly in the kinematical Eq. (1). The boundary condition on the hydrofoil surface is

$$\frac{\partial\varphi}{\partial n} = -U\cos(n, x) \tag{7}$$

where $\frac{\partial\varphi}{\partial n}$ is the normal component of the velocity at the surface. It may also be mentioned here that the real physical situation requires that the disturbance upstream should diminish at a rate which is made to agree with observation.

In order to simplify the problem, it will be assumed that the aspect

ratio

$$\mathcal{R} = \frac{(2b)^2}{S} \tag{8}$$

of the hydrofoil with plan area S and span $2b$ is so large that the whole hydrofoil may be replaced by a lifting line. Then the subsequent analysis can be made similar to the Prandtl wing theory of modern aerodynamics by introducing appropriate modifications of Prandtl's original concept. The fundamental concept is that the circulation distribution along the lifting line which replaces the hydrofoil is the same as the distribution of bound vorticity integrated with respect to the chord of the hydrofoil along the span direction. We then calculate the induced velocity field of this circulation distribution on the lifting line. The characteristic length of this induced velocity field is the span $2b$ or the immersion depth h, whichever is the smaller. Therefore if h is much larger than the representative chord of the hydrofoil, then the characteristic length of the induced velocity field is very much larger than the characteristic length of the local velocity field which is associated with the bound vorticity and the circulation. Then as far as the lift production is concerned, each section of the hydrofoil is effectively surrounded by a stream of infinite extent moving with an effective velocity which is not the free stream velocity U but the sum of U and the induced velocity. We then replace the boundary condition of Eq. (7) by the statement that the circulation of the lifting line at a certain spanwise station is the same as that corresponding to a two-dimensional flow without free surface around the same local section with an effective free stream velocity equal to the sum of U and the induced velocity. Although our argument for the lifting line concept is based upon large aspect ratio $\mathcal{R}$. Experience with airfoils indicates that the theory can be expected to be sufficiently accurate even for $\mathcal{R}$ as low as four.

We shall assume that along the lifting line the distribution of the circulation strength $\Gamma(y)$ is known for given angle of attack and immersion depth. It should be pointed out here that for shallow submergence the effects of the free surface modify considerably the approaching flow velocities and thereby make the pressure and lift distribution different from its corresponding aerodynamic value. Therefore, in our hydrofoil problem the lift distribution is no longer simply proportional to $\Gamma(y)$ as it is in aerodynamic wing theory but is rather a complicate function of $\Gamma(y)$. When $\Gamma(y)$ is given, the lift distribution can be determined as will be shown later. However, the calculation of $\Gamma(y)$ for known lift distribution is more difficult. Strictly

speaking, $\Gamma(y)$ depends not only on the flow velocity, angle of attack and the geometry of the hydrofoil, but also on the depth of submergence h. When h tends to infinity so that this hydrofoil is in a flow of infinite extent, the free surface and wave effects should vanish and $\Gamma(y)$ then tends to its conventional aerodynamic value $\Gamma_\infty(y)$. Since the circulation for equal airfoil and equal effective angle of attack is proportional to the approaching relative flow velocity, we have

$$\Gamma(y) = \left[1 + \frac{\varphi_x(0, y, -h)}{U}\right] \Gamma_\infty(y) . \tag{9}$$

If $\Gamma_\infty(y)$, instead of $\Gamma(y)$ at depth h, were given, Eq. (9) would lead to an integral equation for $\Gamma(y)$ as will be shown later. A similar situation results if one intends to find $\Gamma(y)$ at h for given geometry of the hydrofoil. Therefore we shall first consider the simple case in which $\Gamma(y)$ is assumed to be given.

In this case, the contribution of the lifting line to the perturbation potential can be shown to have the following integral representation (cf. Ref. 22, also cf. Appendix I):

$$\varphi_1 = \frac{\operatorname{sign}(z+h)}{2\pi^2} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\infty} \frac{\sin \lambda x}{\lambda}\, d\lambda \int_0^{\infty} \cos\mu(y-\eta)\, e^{-\sqrt{\lambda^2+\mu^2}\,|z+h|}\, d\mu$$

$$+ \frac{\operatorname{sign}(z+h)}{4\pi} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\infty} \cos\mu(y-\eta)\, e^{-\mu|z+h|}\, d\mu . \tag{10}$$

It may be noted that the above expression satisfies Eq. (1) and has a singularity along the lifting line, a jump of $\Gamma(y)$ across the vortex sheet on the downstream side of the lifting line and is regular elsewhere inside the water. Needless to say, φ_1 has no meaning above the free surface $(z > \zeta)$. It may also be remarked here that some other useful representations for φ_1 can be written as follows (cf. Appendix I) :

$$\varphi_1 = \frac{\operatorname{sign}(z+h)}{4\pi} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\infty} d\xi \int_0^{\infty} e^{-k|z+h|}\, J_0\!\left(k\sqrt{(x-\xi)^2+(y-\eta)^2}\right) k\, dk \tag{10a}$$

or,

$$\varphi_1 = \frac{(z+h)}{4\pi}\int_{-\infty}^{\infty} \frac{\Gamma(\eta)}{(y-\eta)^2+(z+h)^2}\left(1+\frac{x}{\sqrt{x^2+(y-\eta)^2+(z+h)^2}}\right)d\eta, \quad (10b)$$

where J_o denotes the Bessel function of the first kind.

It is convenient to decompose φ into three parts

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 \quad (11)$$

In such a way that the boundary condition of Eq. (6) is decomposed as follows

$$\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial x} = 0 \quad \text{at} \quad z = 0, \quad (12)$$

and

$$U\left(\frac{\partial \varphi_3}{\partial x}\right)_{z=0} = -g\zeta. \quad (13)$$

Physically this decomposition means that φ_3 represents the potential due to wave formation so that φ_3 is the only part of φ which involves the effect of gravity, and it vanishes when the gravitational effect is neglected. In this case the boundary condition Eq. (6) is also relaxed. It does not follow, however, that the disturbance ζ should also vanish because ζ is also influenced by φ_1 and φ_2 through Eq. (5). The remaining part of φ, namely, $\varphi_1 + \varphi_2$, is equivalent to the potential of a biplane system in infinite flow with the upper lifting line of an equal circulation $\Gamma(y)$, rotating in the same sense, distributed along the image points $(z = +h)$ of the real wing. In other words, φ_2 represents the correction to φ due to the effect of the mean free surface $z = 0$. According to this reflection, φ_2 may be written down directly from Eq. (10) by replacing h by $(-h)$. In the range of present interest, we have

$$\varphi_2 = -\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\Gamma(\eta)d\eta\int_0^{\infty}\frac{\sin\lambda x}{\lambda}d\lambda\int_0^{\infty}\cos\mu(y-\eta)e^{\sqrt{\lambda^2+\mu^2}(z-h)}d\mu$$

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\Gamma(\eta)d\eta\int_0^{\infty}\cos\mu(y-\eta)e^{\mu(z-h)}d\mu, \quad \text{for } z \leq 0, \quad (14)$$

which may also be converted into expressions similar to Eqs. (10a) and (10b). It may be noted that this expression is regular everywhere inside the water.

Now it seems plausible to assume that φ_3 and ζ may be represented by the following expressions:

$$\varphi_3 = \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\infty} a(\lambda,\mu)\, \frac{\sin \lambda x}{\lambda}\, d\lambda \int_0^{\infty} \cos \mu(y-\eta)\, e^{\sqrt{\lambda^2+\mu^2}\,(z-h)}\, d\mu$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\infty} \cos \mu(y-\eta)\, e^{\mu(z-h)}\, d\mu\,, \qquad \text{for } z \leq 0\ ;$$

$$\zeta = \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\infty} \beta(\lambda,\mu) \cos \lambda x\, d\lambda \int_0^{\infty} \cos \mu(y-\eta)\, e^{-\sqrt{\lambda^2+\mu^2}\, h}\, d\mu\ ;$$

where $a(\lambda,\mu)$ and $\beta(\lambda,\mu)$ are two functions of μ and λ to be determined by using boundary conditions (5) and (13). Substituting these values into Eqs. (5) and (13), we obtain:

$$a(\lambda,\mu) = -\frac{\kappa}{\pi^2\left(\dfrac{\lambda^2}{\sqrt{\lambda^2+\mu^2}} - \kappa\right)}\,, \qquad \kappa = \frac{g}{U^2}$$

and

$$\beta(\lambda,\mu) = -\frac{U}{g}\, a(\lambda,\mu)\,.$$

These integral representations of φ_3 and ζ can be converted into forms convenient for calculation by introducing the new variables k and θ such that:

$$\lambda = k \cos\theta\,, \qquad \mu = k \sin\theta$$

Then

$$\varphi_3 = -\frac{\kappa}{\pi^2} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{x} d\xi \int_0^{\pi/2} \sec^2\theta\, d\theta \int_0^{\infty} e^{k(z-h)}\, \frac{\cos(k\xi\cos\theta)\cos(k(y-\eta)\sin\theta)}{k - \kappa\sec^2\theta}\, k\, dk$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\infty} e^{\mu(z-h)} \cos \mu(y-\eta)\, d\mu\,, \qquad \text{for } z \leq 0\ ; \qquad (15)$$

$$\zeta = \frac{1}{\pi^2 U}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_0^{\pi/2}\sec^2\theta\,d\theta\int_0^{\infty}e^{-hk}\,\frac{\cos(kx\cos\theta)\cos(k(y-\eta)\sin\theta)}{k-\kappa\sec^2\theta}\,k\,dk \tag{16}$$

It can be seen that this value of ζ given by Eq. (16) is even in x so that the elevation upstream is merely the reflection of that downstream. Therefore it seems necessary to investigate the behavior of ζ for large negative values of x in order to make a comparison with observation. The integral representing ζ as given by Eq. (16) is indeterminate, but it is familiar in plane wave problems that its principal value can be obtained by considering k to be complex and evaluating it by contour integration. The path on the positive real axis should be indented around the point $k = \kappa\sec^2\theta$ which is a simple pole of the integrand. After deforming the contour to the imaginary axis, we obtain:

$$\text{Principal value of } \int_0^{\infty}e^{-hk}\,\frac{\cos(kx\cos\theta)\cos(k(y-\eta)\sin\theta)}{k-\kappa\sec^2\theta}\,k\,dk$$

$$= -(\operatorname{sign} x)\,\pi\kappa\sec^2\theta\, e^{-h\kappa\sec^2\theta}\sin(\kappa x\sec\theta)\cos(\kappa(y-\eta)\sec^2\theta\sin\theta)$$

$$+\int_0^{\infty}e^{-k|x|\cos\theta}\cosh(k(y-\eta)\sin\theta)\,\frac{\kappa\sec^2\theta\cos(kh)+k\sin(kh)}{k^2+\kappa^2\sec^4\theta}\,k\,dk\,. \tag{17}$$

The integration of the last integral in Eq. (17) with respect to θ (cf. Eq. (16)) should only extend over the range for which $|x|\cos\theta \pm (y-\eta)\sin\theta > 0$ for an assigned point (x, y), (cf. App. II for the detail). It is evident from Eq. (17) that although the integral vanishes as $x \to \pm\infty$, the first term remains even for $x \to -\infty$. Then there are surface waves even far ahead of the hydrofoil. This is in contradiction with experience. The paradox can be resolved by observing that there are solutions of the basic flow equation, the free wave solutions, which satisfy the surface pressure condition of Eq. (6). We can then add these solutions to our velocity potential without rendering the solution so far obtained invalid for our problem. The appropriate free wave solution to be added is determined by the condition that the free surface elevation ζ must vanish at large distances ahead of the hydrofoil. Therefore, we have to superimpose on ζ another system of waves

$$\zeta' = -\frac{\kappa}{\pi U}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_0^{\pi/2}\sec^4\theta\, e^{-h\kappa\sec^2\theta}\sin(\kappa x\sec\theta)\cos(\kappa(y-\eta)\sec^2\theta\sin\theta)\,d\theta \tag{18}$$

which is odd in x. In order to remain consistent with all the boundary conditions, we must add to the velocity potential another term, say, φ_4, given by

$$\varphi_4 = -\frac{\kappa}{\pi}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_0^{\pi/2}\sec^3\theta\, e^{\kappa(z-h)\sec^2\theta}\cos(\kappa x\sec\theta)\cos(\kappa(y-\eta)\sec^2\theta\sin\theta)\,d\theta, \quad (\text{for } z \leq 0) \tag{19}$$

which obviously satisfies Eq. (1). This part of the potential may be thought of as a correction term. The same result can be obtained by introducing a fictitious viscous term to the Bernoulli equation (cf. Appendix II, Eq.(18)), a device first discoverec by Lord Rayleigh. The physical significance of this viscous term is, however, not easy to understand. Our argument, even if somewhat lengthy, has its merit.

From the resulting expression of the surface elevation, $\zeta + \zeta'$, it can be seen that the term with the factor $e^{-k|x|\cos\theta}$ in Eq. (17) diminishes exponentially with increasing $|x|$ both in upstream and downstream direction. Hence for large positive values of x, the surface elevation can be approximated by

$$\zeta \sim -\frac{2\kappa}{\pi U}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_0^{\pi/2}\sec^4\theta\, e^{-h\kappa\sec^2\theta}\sin(\kappa x\sec\theta)\cos(\kappa(y-\eta)\sec^2\theta\sin\theta)\,d\theta, \quad (\text{for } x > 0, \text{large}) \tag{20}$$

This relation shows that ζ is analyzed into components of simple waves, where each train of waves (corresponding to one value of θ) propagates on the downstream side of the wave front $x\cos\theta + |y-\eta|\sin\theta = \text{constant}$ (cf. Appendix II) with wave length

$$\lambda = \frac{2\pi}{\kappa\sec^2\theta} = \frac{2\pi U^2}{g}\cos^2\theta \,. \tag{21}$$

This configuration of surface waves resembles that of typical ship waves (cf. Ref. 5, pp. 433-437). To study the behavior of ζ directly behind the hydrofoil as $x \to +\infty$, y finite, we can approximate the integral in Eq. (20)

by applying the method of stationary phase (e.g. cf. Ref. 23, p. 505). Only the result will be given here:

$$\zeta \sim -\frac{1}{U^2}\sqrt{\frac{2g}{\pi x}}\,\sin(\kappa x+\frac{\pi}{4})\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta + O(\frac{1}{x}), \quad \text{as } x \to +\infty,\ y \text{ finite.} \tag{22}$$

Therefore ζ tends to zero like $O(x^{-1/2})$ far downstream ; and at fixed x the effect of free stream velocity is such that ζ is proportional to U^{-1}, noting that in Eq. (22) $\Gamma(y)$ is proportional to U. Eq. (22) shows that as the distance from the hydrofoil is increased only the total circulation on the lifting line matters, the detail of the distribution of circulation is inconsequential. This is, of course, what we would expect from general principles.

It may be also be seen from Eqs. (16) and (18) that there is a local disturbance immediately above the obstacle. As we do not intend to examine the surface wave pattern in detail, the above discussion suffices to describe the general behavior of the surface elevation.

Thus far we have found the complete perturbation potential, namely,

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \tag{23}$$

where φ_1, φ_2, φ_3 and φ_4 are given by Eqs. (10), (14), (15) and (19) respectively.

CALCULATION OF THE DRAG FROM INDUCED VELOCITIES

The total drag, neglecting the part caused by frictional effects, experienced by the hydrofoil may be calculated by using the Kutta-Joukowsky law:

$$D = -\rho\int_{-\infty}^{\infty}\Gamma(y)\,\frac{\partial\varphi(0,y,-h)}{\partial z}\,dy \tag{24}$$

where $-\frac{\partial}{\partial z}\varphi(0, y, -h)$ represents the total induced downwash at the trailing edge of the hydrofoil. If we split D into components, each of which corresponds to the respective component of φ, we have

$$D = D_1 + D_2 + D_3 + D_4 ,$$

where

$$D_n = -\rho \int_{-\infty}^{\infty} \Gamma(y) \frac{\partial}{\partial z} \varphi_n (0, y, -h)\, dy , \quad (n = 1, 2, 3, 4) \tag{25}$$

The calculation of D_1, the induced drag due to the trailing vortices of the hydrofoil itself, is familiar in three-dimensional aerodynamic wing theory and is given by (cf. Ref. 22):

$$D_1 = \frac{\pi \rho}{4} \int_0^{\infty} \left[f(\mu)^2 + g(\mu)^2 \right] \mu \, d\mu \tag{26}$$

where

$$f(\mu) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Gamma(\eta) \cos \mu \eta \, d\eta , \qquad g(\mu) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Gamma(\eta) \sin \mu \eta \, d\eta , \tag{27}$$

$f(\mu)$ and $g(\mu)$ are thus merely the Fourier-coefficients of the circulation distribution $\Gamma(y)$.

For the rest of D, we first calculate $\frac{\partial \varphi_n}{\partial z}$ from Eqs. (14), (15) and (19) and obtain:

$$\frac{\partial}{\partial z} (\varphi_2 + \varphi_3)_{\substack{x=0,\\ z=-h}} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\infty} e^{-2h\mu} \cos(\mu(y-\eta))\, \mu \, d\mu \quad ,$$

$$\frac{\partial \varphi_4}{\partial z}\bigg|_{\substack{x=0,\\ z=-h}} = -\frac{\kappa^2}{\pi} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\pi/2} \sec^5\theta \; e^{-2\kappa h \sec^2\theta} \cos(\kappa(y-\eta)\sec^2\theta \sin\theta)\, d\theta$$

Substitution of these values into Eq. (25) leads to:

$$D_2 + D_3 = -\frac{\pi \rho}{4} \int_0^{\infty} e^{-2h\mu} \left[f(\mu)^2 + g(\mu)^2 \right] \mu \, d\mu \quad , \tag{28}$$

and

$$D_4 = \pi \rho \kappa^2 \int_0^{\pi/2} e^{-2h\kappa\sec^2\theta} \left[f(\kappa \sec^2\theta \sin\theta)^2 + g(\kappa \sec^2\theta \sin\theta)^2 \right] \sec^5\theta \, d\theta \tag{29}$$

where f and g are defined by Eq. (27). The combination of D_2 and D_3 represents the total contribution of the mean free surface effect, which favorably decreases the total drag especially when h is small. As h tends

to zero, $D_2 + D_3$ cancels D_1. The part D_4 represents the wave drag due to the downstream wave formation which results from the gravity effect. It may be remarked here that the wave drag can also be obtained by the method of traveling surface pressure (cf. Appendix III).

From the above result we see that the drag is expressed in terms of the Fourier coefficients, ($f(\mu)$ and $g(\mu)$), of the circulation distribution $\Gamma(y)$. Therefore for a given $\Gamma(y)$, the total drag is completely determined. In most cases, the wing form is symmetric with respect to the central plane $y = 0$, that is,

$$\Gamma(y) = \Gamma(-y) , \tag{30}$$

then it follows from Eq. (27) that g vanishes identically under this condition.

By substituting the relations of Eq. (27) into Eqs. (26), (28) and (29), the drags can be expressed directly in terms of the circulation distribution. Thus

$$D_1 = -\frac{\rho}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\eta)\Gamma(\eta')\frac{d\eta\, d\eta'}{(\eta-\eta')^2} = -\frac{\rho}{4\pi}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_{-\infty}^{\infty}\frac{\frac{d\Gamma}{d\eta'}}{\eta'-\eta}\,d\eta' \tag{26a}$$

$$D_2 + D_3 = -\frac{\rho}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\eta)\Gamma(\eta')\frac{(2h)^2-(\eta-\eta')^2}{\left[(2h)^2+(\eta-\eta')^2\right]^2}\,d\eta\, d\eta' , \tag{28a}$$

and

$$D_4 = \rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\eta)\Gamma(\eta')\,G(\eta-\eta')\,d\eta\, d\eta' \tag{29a}$$

where

$$G(\eta-\eta') = \frac{\kappa^2}{\pi}\int_0^{\pi/2} e^{-2h\kappa\sec^2\theta}\cos\left[(\eta-\eta')\kappa\sec^2\theta\sin\theta\right]\sec^5\theta\, d\theta \tag{29b}$$

Eq. (26a) is the familiar drag formula for the Prandtl lifting line theory. Eq. (29b) shows that $G(\eta-\eta')$ is symmetrical with respect to η and η'.

It is seen that the drag components D_1, D_2 and D_3 are independent of the gravitational effect. They are thus properly called the components of the induced drag. All the drag caused by the presence of surface waves is represented by D_4. D_4 is thus the wave drag. Our calculation does not include, however, the skin friction drag produced by the viscous shearing

stress on the surface of the hydrofoil. This skin friction drag should be added to the drag calculated in this section to obtain the total drag of the hydrofoil.

CALCULATION OF THE TOTAL LIFT

It is well known that the lift exerted on a wing flying in an infinite fluid is given by

$$L_o = \rho U \int_{-\infty}^{\infty} \Gamma(y)\, dy \tag{31}$$

In the hydrofoil problem, however, the approaching stream velocity is influenced by effects of the free surface and wave formations. Hence, the total lift on the hydrofoil will differ from its aerodynamic value L_o by an amount ΔL such that the total lift

$$L = L_o + \Delta L \tag{32}$$

where

$$\Delta L = \rho \int_{-\infty}^{\infty} \Gamma(y) \frac{\partial \varphi(0, y, -h)}{\partial x}\, dy \tag{33}$$

From the expression of φ we find that φ_1 and φ_4 have no contribution to the value of $\frac{\partial \varphi}{\partial x}$ at the hydrofoil. The final result is

$$\frac{\partial \varphi(0, y, -h)}{\partial x} = \left(\frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi_3}{\partial x} \right) \quad \text{at } x = 0, \quad z = -h$$

$$= -\frac{1}{2\pi^2} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\pi/2} d\theta \int_0^{\infty} e^{-2hk} \cos(k(y-\eta)\sin\theta) \frac{k + \kappa \sec^2\theta}{k - \kappa \sec^2\theta}\, k\, dk \tag{34}$$

Substitution of this value into Eq. (33) yields:

$$\Delta L = -\frac{\rho}{2} \int_0^{\infty} e^{-2h\mu} \mu\, d\mu \int_0^{\pi/2} \left[f(\mu \sin\theta)^2 + g(\mu \sin\theta)^2 \right] \frac{\mu + \kappa \sec^2\theta}{\mu - \kappa \sec^2\theta}\, d\theta \tag{35}$$

where f and g are again given by Eq. (27) and g vanishes for symmetrical wings. Equation (35) represents the contribution of free surface effects, which tend to lessen the total lift.

By substituting the relations of Eq. (34) into Eq. (33), a direct expression of ΔL in terms of the circulation distribution $\Gamma(y)$ is obtained:

$$\Delta L = -\frac{\rho}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(\eta)\Gamma(\eta') F(\eta-\eta')\, d\eta\, d\eta' \tag{35a}$$

where

$$F(\eta-\eta') = \frac{1}{\pi^2}\int_0^{\infty} e^{-2h\mu}\, d\mu \int_0^{\pi/2} \cos\left[(\eta-\eta')\mu \sin\theta\right] \frac{\mu+\kappa\sec^2\theta}{\mu-\kappa\sec^2\theta}\, d\theta \tag{35b}$$

Therefore the magnitude of ΔL is in general equal to the magnitude of drag, i.e., of one order smaller than the lift L.

In connection with the present discussion, we examine the effect of immersion depth on the value of $\Gamma(y)$. Substituting Eq. (34) into Eq. (9), we obtain the following relation for a symmetrical hydrofoil,

$$\Gamma(y) = \Gamma_{\infty}(y)\left[1 - \frac{1}{2\pi U}\int_0^{\infty} e^{-2h\mu}\mu\, d\mu \int_0^{\pi/2} f(\mu\sin\theta)\cos(\mu y \sin\theta)\frac{\mu+\kappa\sec^2\theta}{\mu-\kappa\sec^2\theta}\right] d\theta \tag{36}$$

where $\Gamma_{\infty}(y)$ is the value of $\Gamma(y)$ for the same wing and same effective angle of attack as $h \to \infty$. If $\Gamma(y)$ at depth h is given, then the aerodynamic value of the circulation of the same wing, $\Gamma_{\infty}(y)$, can be calculated from Eq. (36). On the other hand, if only $\Gamma_{\infty}(y)$ is known, then Eq. (36) provides an integral equation for $\Gamma(y)$ since f also depends on $\Gamma(y)$. However, for moderate values of h, $\Gamma(y)$ and $\Gamma_{\infty}(y)$ are approximately equal as will be shown later.

GEOMETRY OF THE HYDROFOIL; EFFECTIVE ANGLE OF ATTACK

Take a strip of the hydrofoil of width (dy) in the spanwise direction and with chord $c(y)$ located at y. According to the basic concepts of the lifting line theory explained in the section "General Formulation of the Problem",

the circulation $\Gamma(y)$ around the lifting line at y is the same as the circulation around the corresponding hydrofoil section. It then follows that the lift on this lifting line segment is the same as the lift on this hydrofoil section, integrated with respect to the chord of the hydrofoil. The local flow around this hydrofoil section in a plane y = constant may be then considered as a two-dimensional flow around the same hydrofoil section, with free stream velocity $(U + u(y),\ w(y))$ and without free water surface. Therefore the local lift coefficient $C_\ell(y)$ and drag coefficient $C_d(y)$ per unit chord length can be defined by

$$\frac{1}{2}\rho\left[U+u(y)\right]^2 c(y)\, C_\ell(y)\, dy = \rho\left[U+u(y)\right]\Gamma(y)\, dy$$

$$\frac{1}{2}\rho\left[U+u(y)\right]^2 c(y)\, C_d(y)\, dy = \rho\, w(y)\, \Gamma(y)\, dy$$

where $u(y)$ and $w(y)$ denote the induced velocities $\frac{\partial\varphi}{\partial x}$ and $(-\frac{\partial\varphi}{\partial z})$ respectively at the hydrofoil. The above relations then give

$$\Gamma(y) = \frac{1}{2}\left[U+u(y)\right] c(y)\ C_\ell(y) \tag{37}$$

and

$$\epsilon(y) = \frac{C_d}{C_\ell} = \frac{w(y)}{U+u(y)} \tag{38}$$

where $\epsilon(y)$ is defined by the above equation as the downwash angle. Eq. (38) indicates an important feature of the local flow that the actual absolute angle of attack α_a, measured from the free stream to the zero-lift direction, is modified to an effective angle of attack α_e, measured from the approaching stream to the zero-lift direction on account of the local perturbed velocity as shown in Fig. 2. The relation between α_a and α_e is given by

$$\alpha_e = \alpha_a - \epsilon(y) \tag{39}$$

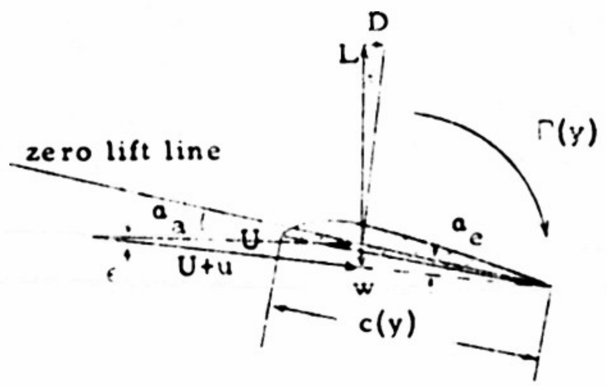

Fig. 2 - Downwash and effective angle of attack.

In this way we can separate the three-dimensional effects and two-dimensional effects by taking the local sectional characteristics of the hydrofoil at depth h and at absolute incidence α_a to be the same characteristics of the section of a two-dimensional hydrofoil submerged at infinite depth and sustained at an absolute incidence equal to $\alpha_e = \alpha_a - \epsilon(y)$. Referring to these two incidences, $C_\ell(y)$ is approximately proportional to α_a or α_e for most of the usual profiles at small incidences (Ref. 17 and 21). Moreover, when α_a vanishes so do α_e and $\epsilon(y)$. Hence we may put

$$C_\ell(y) = a_a \alpha_a = a_e \alpha_e . \tag{40}$$

In general, all a_a, α_a, a_e and α_e may be functions of y. In particular, α_a is constant for wings with no geometric twist, and in addition with the same profile along the span, a_e is constant. Combining Eqs. (37) - (40), we have

$$\Gamma(y) = \frac{U\, c(y)\, a_e}{2} \left[\alpha_a \left(1 + \frac{u(y)}{U}\right) - \frac{w(y)}{U} \right] . \tag{41}$$

For given $\Gamma(y)$, $u(y)$ and $w(y)$ are determined, then this relation gives the value of the chord length $c(y)$ except for a proportionality constant. In particular,

$$\Gamma(0) = \frac{U\, c_o\, a_e}{2} \left[\alpha_a \left(1 + \frac{u(0)}{U}\right) - \frac{w(0)}{U} \right] \tag{41a}$$

which gives the relation between $\Gamma(0)$ and c_o, the chord at the central section.

FORMULATION OF PROBLEM WITH SPECIFIED GEOMETRY

The problem is for given geometry of the hydrofoil, that is, given b, $c(y)$, $\alpha_a(y)$, a_e and h, to find $\Gamma(y)$, C_L and C_D. The relation between the given quantities and the unknown $\Gamma(y)$ was approximated in the section, "Geometry of the Hydrofoil; Effective Angle of Attack", to be

$$\frac{\Gamma(y)}{\frac{1}{2} U\, c(y)\, a_e} = \alpha_a \left(1 + \frac{u(y)}{U}\right) - \frac{w(y)}{U} , \tag{42}$$

where from Eq. (34) and the section "Calculation of the Drag from Induced Velocities",

$$u(y) = -\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_0^{\pi/2}d\theta\int_0^{\infty}e^{-2h\mu}\cos\left[\mu(y-\eta)\sin\theta\right]\frac{\mu+\kappa\sec^2\theta}{\mu-\kappa\sec^2\theta}\,\mu\,d\mu\,, \tag{43}$$

$$w(y) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_0^{\infty}(1-e^{-2h\mu})\cos\left[\mu(y-\eta)\right]\mu\,d\mu$$

$$+\frac{\kappa^2}{\pi}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_0^{\pi/2}\sec^5\theta\,e^{-2h\kappa\sec^2\theta}\cos\left[\kappa(y-\eta)\sec^2\theta\sin\theta\right]d\theta \tag{44}$$

One way to solve this integral equation is by using a method similar to that of I. Lotz[27] in aerodynamic wing theory. This method consists in assuming for the circulation $\Gamma(y)$ at distance y from the plane of symmetry the formula

$$\Gamma(y) = 4bU\sum_{n=0}^{\infty}A_{2n+1}\sin(2n+1)\phi\,, \qquad y = b\cos\phi\,; \tag{45}$$

$$= 0\,, \qquad |y| \geq b$$

so that $\Gamma(y) = \Gamma(-y)$ and

$$\Gamma(0) = 4bU\sum_{n=0}^{\infty}(-)^n A_{2n+1}\,. \tag{45a}$$

The coefficients A_{2n+1} are to be determined by using the condition (42). The first term of the Fourier series in Eq. (45), namely,

$$\Gamma_1(y) = 4bU A_1\sin\phi = 4bUA_1\sqrt{1-\frac{y^2}{b^2}}\,, \tag{45b}$$

represents an elliptical distribution of the circulation. The rest of the terms may be regarded as a measure of the deviation between the actual distribution and the elliptical.

Substituting Eq. (45) in (43) and integrating by parts with respect to η, we obtain (cf. Appendix IV, A)

$$u(y) = -U\sum_{n=0}^{\infty}(-)^n(2n+1)A_{2n+1}\,\mathcal{U}_{2n+1}(\phi) \tag{46}$$

where

$$\mathcal{U}_{2n+1}(\phi) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sin\theta}\int_0^{\infty} e^{-2\lambda t}\, \frac{t + b\kappa\sec^2\theta}{t - b\kappa\sec^2\theta}\, J_{2n+1}(t\sin\theta)\cos(t\cos\phi\,\sin\theta)\, dt, \tag{46a}$$

$$\lambda \equiv \frac{h}{b} = \text{depth} - \frac{1}{2}\text{ span ratio} . \tag{46b}$$

In a similar manner, substitution of (45) in (44) gives (cf. Appendix IV, B)

$$w(y) = \frac{U}{\sin\phi}\sum_{n=0}^{\infty}(2n+1)\; A_{2n+1}\mathcal{W}_{2n+1}(\phi) \tag{47}$$

where

$$\mathcal{W}_{2n+1}(\phi) = \sin(2n+1)\phi + (-)^{n+1}(\sin\phi)\,\mathrm{R}\ell\, \frac{\left[\sqrt{1+(2\lambda - i\cos\phi)^2} - (2\lambda - i\cos\phi)\right]^{2n+1}}{\sqrt{1+(2\lambda - i\cos\phi)^2}}$$

$$+ (-)^n 4b\kappa\sin\phi\int_0^{\pi/2}\left[e^{-2h\kappa\sec^2\theta}J_{2n+1}(b\kappa\sec^2\theta\sin\theta)\cos(b\kappa\cos\phi\sec^2\theta\sin\theta)\,\frac{\sec^3\theta}{\sin\theta}\right]d\theta \tag{47a}$$

For symmetrical wing we require that both the chord $c(y)$ and the incidence $a_a(y)$ are even in y, hence we may expand the following quantities into Fourier series with known coefficients

$$\frac{c_o\sin\phi}{c(y)} = \sum_{n=0}^{\infty} C_{2n}\cos 2n\phi \quad , \tag{48}$$

$$a_a(y)\sin\phi = \sum_{n=0}^{\infty} B_{2n}\cos 2n\phi \quad . \tag{49}$$

Substituting Eqs. (45) - (49) into Eq. (42), we get

$$\frac{8b}{c_o a_e}\left[\sum_{n=0}^{\infty}C_{2n}\cos 2n\phi\right]\left[\sum_{n=0}^{\infty}A_{2n+1}\sin(2n+1)\phi\right]$$

$$= \left[1 - \sum_{n=0}^{\infty}(-)^n(2n+1)A_{2n+1}\mathcal{U}_{2n+1}(\phi)\right]\left[\sum_{n=0}^{\infty}B_{2n}\cos 2n\phi\right] - \sum_{n=0}^{\infty}(2n+1)A_{2n+1}\mathcal{W}_{2n+1}(\phi) \tag{50}$$

where $\mathcal{U}_{2n+1}$ and $\mathcal{W}_{2n+1}$ are given by Eqs. (46a) and (47a). To determine the coefficients A_{2n+1} from Eq. (50) would necessitate the expansion of the

left side, and both $u_{2n+1}(\phi)$ and $w_{2n+1}(\phi)$, into a Fourier series in ϕ, thus leading to infinitely many linear equations, in infinitely many unknowns. The solution to this problem is obviously a difficult one. However, a very good approximation can be obtained by using a method for practical calculation on wings of finite span, due to H. Glauert[28]. Replace the infinite series representation of $\Gamma(y)$ given by Eq. (45) by an approximate value in terms of a finite series of, say, $m+1$ terms,

$$\Gamma(y) = 4bU \sum_{n=0}^{m} A_{2n+1} \sin(2n+1)\phi , \quad y = b\cos\phi . \qquad (51)$$

This expression of $\Gamma(y)$ then reduces the condition of Eq. (42) to

$$\frac{8b}{a_e}\left[\frac{\sin\phi}{c(y)}\right]\left[\sum_{n=0}^{m} A_{2n+1}\sin(2n+1)\phi\right] = \left[1 - \sum_{n=0}^{m}(-)^n(2n+1)A_{2n+1}u_{2n+1}(\phi)\right](\alpha_a(y)\sin\phi)$$
$$- \sum_{n=0}^{m}(2n+1)A_{2n+1}w_{2n+1}(\phi) . \qquad (52)$$

This equation cannot be satisfied identically for all values of ϕ. However, if $(m+1)$ particular values are suitably chosen for ϕ, we get $m+1$ linear equations from which the $(m+1)$ coefficients, A_1, A_3, - - A_{2m+1}, can be determined. The values of these A_{2n+1} so obtained will satisfy Eq. (52), not identically, but only at the selected points. In general, the first four coefficients usually give a sufficiently accurate result. The detail of such calculations will not be given here; however, the calculation of the first order term is quite similar to that of a specific example of the direct problem discussed in the section "Example - Elliptical Distribution of the Circulation Strength".

Having obtained the value of $\Gamma(y)$ for this indirect problem, the calculation of the lift and the drag is then the same as that of the direct problem discussed in section "Calculation of the Drag from Induced Velocities" and section "Calculation of the Total Lift". The results can be directly written down by ascribing to $f(\mu)$ and $g(\mu)$ the following value:

$$f(\mu) = \frac{1}{\pi}\int_{-b}^{b}\Gamma(\eta)\cos\mu\eta\, d\eta = -\frac{1}{\pi\mu}\int_{-b}^{b}\sin\mu\eta\,\frac{d\Gamma(\eta)}{d\eta}\,d\eta$$

$$= \frac{4bU}{\pi\mu}\sum_{n=0}^{\infty}(2n+1)A_{2n+1}\int_{0}^{\pi}\sin(\mu b\cos\psi)\cos(2n+1)\psi\, d\psi$$

$$= \frac{4bU}{\mu}\sum_{n=0}^{\infty}(-)^n(2n+1)A_{2n+1}J_{2n+1}(b\mu) \qquad (53a)$$

and

$$g(\mu) = 0 \tag{53b}$$

MINIMUM DRAG FOR GIVEN LIFT

We shall consider a hydrofoil of span $2b$ so that $\Gamma(y) = 0$ for $|y| \geqslant b$. The problem is to find a distribution $\Gamma(y)$ such that the drag D is minimum under the condition of constant lift L. By the method of undetermined multipliers the requirement is

$$\delta D - \lambda_o \delta L = 0 \tag{54}$$

where the variation is carried out such that $\delta\Gamma(y) = 0$ at $y = \pm b$. Applying the above variation to the general expressions for D and L given by Eqs. (26a)-(29b), (31), (35a), and (35b), using the relation that $\delta\left[\Gamma(\eta)\Gamma(\eta')\right] = \Gamma(\eta)\delta\Gamma(\eta') + \Gamma(\eta')\delta\Gamma(\eta)$ and noting that both $F(\eta-\eta')$ and $G(\eta-\eta')$ defined respectively by Eqs. (35b) and (29b) are symmetric with respect to $(\eta-\eta')$, we obtain

$$\int_{-b}^{b}\left[-2\,\frac{\partial\varphi(0,y,-h)}{\partial z} - \lambda_o\left(U + 2\,\frac{\partial\varphi(0,y,-h)}{\partial x}\right)\right]\delta\Gamma(y)\,dy = 0 \,.$$

Since $\delta\Gamma(y)$ is otherwise arbitrary, the quantity inside the bracket must vanish identically. Or, using the present notation, we have

$$2\,w(y) - \lambda_o\left[U + 2\,u(y)\right] = 0 \,. \tag{55}$$

The constant multiplier λ_o determined by Eq. (55) equals approximately twice the value of the downwash angle ϵ for not too shallow submergences; more precisely, the difference $\epsilon - \frac{\lambda_o}{2}$ is a second order small quantity given by

$$\epsilon - \frac{\lambda_o}{2} = \frac{u(y)\,w(y)}{\left[U + u(y)\right]\left[U + 2u(y)\right]} \,.$$

Therefore, the condition for the minimum total drag, accounting for all causes and holding lift constant, is that the total downwash angle must be constant, along the span, up to the first order term. When depth of submergence becomes infinite, $u(y)$ tends to zero, the above condition is reduced

to the requirement of constant downwash which is in agreement with aerodynamic wing theory.

Substituting the explicit expressions of the induced velocities $w(y)$ and $u(y)$ given by Eqs. (43) and (44) into Eq. (55), we obtain the following integral equation for $\Gamma(y)$:

$$\frac{1}{2\pi}\int_{-b}^{b}\Gamma(\eta)\,d\eta\left[\int_{0}^{\infty}(1-e^{-2h\mu})\cos\mu(y-\eta)\mu\,d\mu \right.$$

$$\left. + 4\kappa^2\int_{0}^{\pi/2} e^{-2\kappa h\sec^2\theta}\cos(\kappa(y-\eta)\sec^2\theta\sin\theta)\sec^5\theta\,d\theta\right]$$

$$= \lambda_o\left[U - \frac{1}{\pi^2}\int_{-b}^{b}\Gamma(\eta)\,d\eta\int_{0}^{\infty}e^{-2h\mu}\mu\,d\mu\int_{0}^{\pi/2}\cos(\mu(y-\eta)\sin\theta)\frac{\mu+\kappa\sec^2\theta}{\mu-\kappa\sec^2\theta}\,d\theta\right] \qquad (55a)$$

To solve this equation a practical method of approximation similar to that used in solving Eq. (42) of the indirect problem may also be applied here. Assuming that the exact solution of Eq. (55a) may be approximated by the expression given by Eq. (51), Eq. (55) can be reduced to the form

$$\sum_{n=o}^{m}(2n+1)\,A_{2n+1}w_{2n+1}(\emptyset) = \lambda_o\sin\emptyset\left[\frac{1}{2} - \sum_{n=o}^{m}(-)^n(2n+1)\,A_{2n+1}u_{2n+1}(\emptyset)\right] \qquad (56)$$

where u_{2n+1} and w_{2n+1} are given by Eqs. (46a) and (47a) respectively. From this equation the first $(m+1)$ coefficients A may then be determined for $(m+1)$ particularly selected values of $\emptyset$. It should be remarked here again that the solution so obtained cannot satisfy Eq. (56) identically in $\emptyset$, but nevertheless gives a good approximation. The estimation of the deviation of this solution from the elliptical distribution, which is the solution of the corresponding problem in aerodynamic wing theory, will be made in the specific example discussed below.

EXAMPLE - ELLIPTICAL DISTRIBUTION OF THE CIRCULATION STRENGTH

As shown in the indirect problem that the first term in the assumed expansion of $\Gamma(y)$ given by Eq. (45) represents the so-called elliptical distribution, it would be of interest to consider a direct problem of a hydrofoil of span $2b$ immersed at a fixed depth h with an elliptical distribution of $\Gamma(y)$, that is,

$$\begin{aligned} \Gamma(y) &= \Gamma_o \sqrt{1 - \frac{y^2}{b^2}}\,, \qquad |y| \leqslant b\,, \\ &= 0\,, \qquad\qquad\qquad\quad |y| \geqslant b\,. \end{aligned} \tag{57}$$

Although for a given hydrofoil of fixed geometrical form this distribution cannot be held for different values of h, nevertheless it gives interesting results. In this problem we shall further define the Froude number of the motion by

$$\sigma = \frac{1}{h\kappa} = \frac{U^2}{gh} \tag{58a}$$

which is one of important parameters of this problem. We shall mostly be concerned with large values of σ, corresponding to shallow submergences. For instance, for U equal to 80 ft/sec and h equal to 4 ft, σ is approximately 50. We shall further denote

$$\beta = \frac{1}{b\kappa} = \frac{U^2}{gb}\,, \qquad \frac{\beta}{\sigma} = \frac{h}{b} = \lambda\,. \tag{58b}$$

β is in general also very large in our velocity range of interest.

For this distribution of $\Gamma(y)$ we find, from Eq. (27),

$$f(\mu) = \frac{2\Gamma_o}{\pi}\int_o^b \sqrt{1 - \frac{y^2}{b^2}}\cos(\mu y)\,dy = \Gamma_o \frac{J_1(\mu b)}{\mu}\,, \qquad g(\mu) = 0\,. \tag{59}$$

a. <u>Calculation of the Drag</u>

Substituting Eq. (59) into Eq. (26), we have (cf. Ref. 24, p. 405):

$$D_1 = \frac{\pi\rho\Gamma_o^2}{4}\int_o^\infty \frac{J_1^2(\mu b)}{\mu}\,d\mu = \frac{\pi\rho\Gamma_o^2}{8}\;; \tag{60}$$

and from Eq. (28) we obtain (cf. App. IV, C)

$$D_2 + D_3 = -\frac{\pi \rho \Gamma_o^2}{4} \int_0^{\infty} e^{-2h\mu} \frac{J_1^2(\mu b)}{\mu} d\mu$$

$$= -\frac{\pi \rho \Gamma_o^2}{8} \left\{ 1 - \frac{4}{\pi} \lambda \sqrt{1+\lambda^2} \left[K\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - E\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \right] \right\} \qquad (61)$$

where $K(k)$ and $E(k)$ denote the complete elliptic integral of the first and second kind respectively and λ is the depth - $\frac{1}{2}$ span ratio defined by Eq. (58b). Using the series expansions and the asymptotic expansions of K and E (cf. Ref. 26, p. 73), we find the behavior of $(D_2 + D_3)$ for small and large values of λ respectively as follows:

$$D_2 + D_3 \cong -\frac{\pi \rho \Gamma_o^2}{8} \left\{ 1 - \frac{4}{\pi} \frac{\lambda}{\sqrt{1+\lambda^2}} \left[\log \frac{4\sqrt{1+\lambda^2}}{\lambda} - 1 + \frac{3}{4} \frac{\lambda^2}{\sqrt{1+\lambda^2}} \left(\log \frac{4\sqrt{1+\lambda^2}}{\lambda} - \frac{4}{3}\right) + O\left(\frac{\lambda^4}{(1+\lambda^2)^2} \log \frac{4\sqrt{1+\lambda^2}}{\lambda}\right) \right] \right\}$$

as $h \to 0$; (62a)

$$D_2 + D_3 = -\frac{\pi \rho \Gamma_o^2}{64 \lambda^2} \left\{ 1 - \frac{1}{4\lambda^2} + O\left(\frac{1}{\lambda^4}\right) \right\} \qquad \text{as } h \to \infty . \qquad (62b)$$

As $h \to \infty$, the drag due to the surface effect diminishes like $\frac{1}{h^2}$. When the hydrofoil becomes a planing surface, i.e. $h \to 0$, we find that $(D_2 + D_3)$ tends to a finite value $-\pi \rho \Gamma_o^2/8$ which cancels D_1, and consequently, the drag is then solely due to the wave effect.

To find the wave drag, we substitute Eq. (59) into Eq. (29) and obtain the following integral*:

$$D_4 = \pi \rho \Gamma_o^2 \int_0^{\pi/2} e^{-\frac{2}{\sigma} \sec^2\theta} \left[\frac{J_1(\frac{1}{\beta} \sec^2\theta \sin\theta)}{\sec^2\theta \sin\theta} \right]^2 \sec^5\theta \, d\theta \qquad (63)$$

---

* From the integral representation of the wave drag given by Eq. (63), we note that

$$D_4 = \rho \Gamma_o^2 f(\sigma, \lambda), \quad \text{or,} \quad \frac{D_4}{\rho U^2 c_o^2} = f(\sigma, \lambda)$$

which can be directly derived from dimensional analysis (cf. Ref. 5, p. 438). This is a useful non-dimensional form for experimental purpose.

For general values of the Froude number σ, the above integral may be converted into an infinite series, each term of which contains modified Bessel functions of the second kind, (cf. App. IV, D), as follows:

$$D_4 = \frac{\rho\Gamma_o^2}{4} \frac{e^{-1/\sigma}}{\beta^2} \sum_{n=o}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!(n+1)!(n+2)!} \left(\frac{\sigma}{2\beta^2}\right)^n \left\{ K_n(\frac{1}{\sigma}) + \left[1+\sigma(n+\frac{1}{2})\right] K_{n+1}(\frac{1}{\sigma}) \right\} \quad (64)$$

For very small values of σ ($\sim$ large h, or λ), we can use the asymptotic expansion of $K_n(\frac{1}{\sigma})$,

$$K_n(\frac{1}{\sigma}) \cong \sqrt{\frac{\pi\sigma}{2}}\, e^{-1/\sigma} \left[1 + \frac{4n^2-1^2}{1!\ 8}\sigma + \frac{(4n^2-1)(4n^2-3^2)}{2!\ (8)^2}\sigma^2 + \ldots\ldots \right]. \quad (65)$$

So we see that in this case the series in Eq. (64) converges very fast and the final result may be expressed asymptotically by

$$D_4 \cong \frac{\pi\rho\Gamma_o^2}{8} \frac{e^{-2/\sigma}}{\beta^2} \left(\frac{\pi\sigma}{2}\right)^{1/2} \left[1 + \frac{3}{8}\left(1 - \frac{1}{6\beta^2}\right)\sigma + 0(\sigma^2) \right], \qquad \text{as } \sigma \to 0, \quad (66)$$

which diminishes exponentially with increasing h. As a remark, the above asymptotic expression of D_4 for σ small can also be obtained by evaluating the integral of Eq. (63) in a complex θ plane and applying the method of steepest descent (e.g. cf. Ref. 23, p. 504).

For large values of σ ($\sim$ small h), $K_n(\frac{1}{\sigma})$ has the following expansion:

$$K_o(\frac{1}{\sigma}) = \log(2\sigma)\cdot I_o(\frac{1}{\sigma}) + \sum_{m=o}^{\infty} \frac{(2\sigma)^{-2m}}{(m!)^2}\psi(m+1),$$

$$K_n(\frac{1}{\sigma}) = \frac{1}{2}\sum_{m=o}^{n-1} \frac{(-)^m(n-m-1)!}{m!}(2\sigma)^{n-2m} \quad (67)$$

$$+ (-)^n \sum_{m=o}^{\infty} \frac{(2\sigma)^{-n-2m}}{m!(n+m)!}\left[\log(2\sigma) + \frac{1}{2}\psi(m+1) + \frac{1}{2}\psi(n+m+1)\right]. \qquad (n \geq 1)$$

It can be seen that the most important contribution to the sum of the series in Eq. (64) comes from the term $\frac{1}{2}n!\,(2\sigma)^{n+1}$ of the expansion of $K_{n+1}(\frac{1}{\sigma})$. As shown in Appendix IV, E, this term results in a series which converges only for $\lambda \geq 1$ and represents, in this region, an analytic function of λ. Consequently the value of the wave drag D_4 for shallow submergences

$(\sim\lambda \sim 1)$ may be obtained by some analytical continuation of this function from the region $\lambda \geq 1$ to the whole region of physical interest, namely, $0 \leq \lambda < \infty$ (cf. App. IV, E). The final result is

$$D_4 \cong \frac{\pi\rho\Gamma_o^2}{4}\left\{ 1 - \frac{4}{\pi}\lambda\sqrt{1+\lambda^2}\left[K\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - E\left(\frac{1}{\sqrt{1+\lambda^2}}\right)\right] + \frac{4}{3\beta}\left[\frac{2}{\pi}(1+\lambda^2)^{3/2} E\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - \frac{3}{2}\lambda - \lambda^2\sqrt{1+\lambda^2}\, F\left(-\frac{1}{2}, \frac{3}{2}; 1; \frac{1}{1+\lambda^2}\right)\right] + \frac{3}{16\beta^2}\log\beta + 0\left(\frac{1}{\beta^2}, \frac{1}{\sigma}\right)\right\}, \quad \text{as } \sigma \to \infty, \tag{68}$$

It is of interest to note that when the hydrofoil is close to the surface, the wave drag is rather similar in behavior but of the opposite sign and numerically almost twice of the drag due to the surface effect (cf. Eq. (61)). However, it should be emphasized here that even $\Gamma(y)/\Gamma_o$ were the same function of y for all values of λ, Γ_o still depend on λ, especially for λ small, (cf. Eqs. (9) and (41a)). Consequently the actual dependence of the induced and wave drag on λ for a given hydrofoil at fixed geometric incidence can only be determined after the function $\Gamma_o(\lambda)$ is found later in this section, part (d).

It is also of interest to compare, when σ is small, the first term in Eq. (64),

$$D' = \frac{\pi\rho\Gamma_o^2}{16}\left(\frac{g^2 b^2}{U^4}\right) e^{-1/\sigma}\left[K_o\left(\frac{1}{\sigma}\right) + \left(1 + \frac{\sigma}{2}\right) K_1\left(\frac{1}{\sigma}\right)\right],$$

with the wave resistance on a moving sphere of radius r given approximately by (cf. Ref. 5, p. 437, also Ref. 6):

$$R = \pi\rho\left(\frac{g^2 r^3}{U^3}\right)^3 e^{-1/\sigma}\left[K_o\left(\frac{1}{\sigma}\right) + \left(1 + \frac{\sigma}{2}\right) K_1\left(\frac{1}{\sigma}\right)\right], \quad \left(\sigma = \frac{U^2}{g h}\right).$$

Except for a proportionality constant which depends on the size of the obstacle, they have the same dependence on σ. Evidently an observer on water surface cannot distinguish bodies of different forms if they move at sufficiently great depth.

b. Calculation of the Lift

The total lift experienced by the hydrofoil can be obtained, by substituting Eq. (59) into Eqs. (31)-(35), as follows:

$$L = L_o + \Delta L_1 + \Delta L_2 \tag{69}$$

where

$$L_o = \rho U \Gamma_o \int_{-b}^{b} \sqrt{1 - \frac{y^2}{b^2}}\, dy = \frac{\pi}{2} \rho U b \Gamma_o , \tag{70}$$

$$\Delta L_1 = \frac{-\rho \Gamma_o^2}{2} \int_0^{\pi/2} \frac{d\theta}{\sin^2\theta} \int_0^{\infty} e^{-2h\mu} J_1^2(b\mu \sin\theta) \frac{d\mu}{\mu} , \tag{71}$$

$$\Delta L_2 = -\kappa \rho \Gamma_o^2 \int_0^{\pi/2} \frac{d\theta}{\sin^2\theta} \int_0^{\infty} e^{-2h\mu} J_1^2(b\mu \sin\theta) \frac{\sec^2\theta\, d\mu}{\mu(\mu - \kappa \sec^2\theta)} . \tag{72}$$

It can be seen that L_o is the aerodynamic value of the lift, ΔL_1 is independent of the gravitational effect and thus represents the correction due to the mean free surface. All the lift caused by the surface wave effects is represented by ΔL_2. For small incidence angles, Γ_o is approximately proportional to the incidence angle (cf. Eq. (41a)), hence relative to L_o, ΔL and D are second order small quantities.

The integral representing ΔL_1 contains the same integral given in Eq. (61) (cf. also App. IV, C), the result of which may be then applied here. After this substitution, integrating by parts with respect to θ, we obtain a simpler representation for ΔL_1 (cf. App. IV, F) :

$$\Delta L_1 = -\frac{\rho \Gamma_o^2}{\pi \lambda} \int_0^{\chi} \sqrt{\frac{1 - (\frac{k}{\chi})^2}{1 - k^2}}\, \mathcal{C}(k)\, dk \tag{73}$$

where $\mathcal{C}(k)$ is a derived complete elliptic integral (cf. Ref. 25, p. 73) defined by

$$\mathcal{C}(k) = \int_0^{\pi/2} \frac{\sin^2\phi \cos^2\phi}{[1 - k^2 \sin^2\phi]^{3/2}}\, d\phi \tag{73a}$$

and

$$\chi = \frac{1}{\sqrt{1+\lambda^2}} \tag{73b}$$

For $0 < h < \infty$ we have the corresponding values of λ and χ as $0 < \lambda < \infty$ and $1 > \chi > 0$. When $\lambda \to 0$ $(\chi \to 1)$, the integral in Eq. (73) tends to a definite value, namely

$$\lim_{\chi \to 1-0} \int_0^{\chi} \mathcal{C}(k) \sqrt{\frac{1-(\frac{k}{\chi})^2}{1-k^2}}\, dk = \int_0^1 \mathcal{C}(k)\, dk$$

$$= \frac{1}{3}\Big[k\,B(k) - (1-k^2)\,k\,\mathcal{C}(k)\Big]_0^1 = \frac{1}{3} \tag{74}$$

where $B(k)$ is another derived complete elliptic integral (cf. Ref. 25, p. 78) defined by

$$B(k) = \frac{1}{k^2}\Big[E(k) - (1-k^2)\,K(k)\Big]\,. \tag{74a}$$

Hence Eq. (73) indicates that ΔL_1 tends to ∞ like $\frac{1}{\lambda}$ as $\lambda \to 0$, noting that Γ_o^2 does not vanish, as will be shown later. This fact is not surprising because it is known that as the hydrofoil approaches the planning condition, the total lift drops to almost half of its aerodynamic value L_o (e.g. cf. Ref. 14). The divergence of the integral in Eq. (71) simply implies that this second order small quantity will grow so large that it will modify the value of the first order quantity L_o.

For λ not too small ($\sim \chi$ not too close to 1), the integral in Eq. (73) can be calculated by using the known expansion of $\mathcal{C}(k)$, (cf. Ref. 25, p. 73), given as follows:

$$\mathcal{C}(k) = \frac{\pi}{16}\left[1 + 6\,\frac{k^2}{8} + \frac{75}{2}\left(\frac{k^2}{8}\right)^2 + 245\left(\frac{k^2}{8}\right)^3 + 0(k^8)\right]. \tag{75}$$

Substituting this expansion into Eq. (73), then applying the transformation $k^2 = \chi^2 t$ and integrating termwise, we obtain (cf. App. IV, G) :

$$\Delta L_1 = -\frac{\pi\rho\Gamma_o^2}{64\lambda\sqrt{1+\lambda^2}}\left[1 + \frac{5}{16}\frac{1}{(1+\lambda^2)} + \frac{181}{256}\frac{1}{(1+\lambda^2)^2} + \frac{3015}{8^5}\frac{1}{(1+\lambda^2)^3} + 0\left(\frac{1}{(1+\lambda^2)^4}\right)\right] \tag{76}$$

This formula should be good for $\lambda > 1$ (or $h > b$). For β close to 1 ($h \approx 0$), the above series converges slowly; hence we have to resort to some other approximation in order to facilitate calculation. For this case, we define

$$\chi'^2 = 1 - \chi^2 , \quad \chi'^2 << 1 , \quad \chi'^2 = \frac{\lambda^2}{1+\lambda^2} = \lambda^2 + 0(\lambda^4) . \tag{77}$$

Then

$$\sqrt{\frac{1-(\frac{k}{\chi})^2}{1-k^2}} = 1 - \frac{1}{2}\chi'^2 \left(\frac{k^2}{1-k^2}\right) + 0(\chi'^4) \ ;$$

and

$$\Delta L_1 = -\frac{\rho\Gamma_o^2}{\pi\lambda}\left[\int_0^{\chi} \mathcal{C}(k)\,dk - \frac{1}{2}\chi'^2\int_0^{\chi}\frac{k^2}{1-k^2}\,\mathcal{C}(k)\,dk + 0(\chi'^4)\right].$$

Now the first integral can be integrated,

$$\int_0^{\chi} \mathcal{C}(k)\,dk = \frac{1}{3}\left[\chi\, B(\chi) - \chi\chi'^2\, \mathcal{C}(\chi)\right],$$

and the second integral may be approximated by

$$\int_0^{\chi}\frac{k^2}{1-k^2}\,\mathcal{C}(k)\,dk = -\frac{1}{4}\log\sqrt{1-\chi^2} = -\frac{1}{4}\log\frac{\lambda}{\sqrt{1+\lambda^2}} .$$

Finally we have

$$\Delta L_1 \cong -\frac{\rho\Gamma_o^2}{3\pi\lambda}\left\{\frac{1}{\sqrt{1+\lambda^2}}\, B\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - \frac{\lambda^2}{\sqrt{1+\lambda^2}}\left[\frac{1}{\sqrt{1+\lambda^2}}\, \mathcal{C}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) + \frac{3}{8}\log\frac{\sqrt{1+\lambda^2}}{\lambda}\right]\right\}$$

$$+ 0(\lambda^4 \log\lambda) , \tag{78}$$

and using the known expansion of $B(\chi)$ and $\mathcal{C}(\chi)$ as $\chi \to 1$, we obtain

$$\Delta L_1 \cong -\frac{\rho\Gamma_o^2}{3\pi\lambda}\left\{1 - 3\lambda^2\left[\log 2 - \frac{3}{4} + \frac{5}{8}\log\frac{\sqrt{1+\lambda^2}}{\lambda}\right]\right\} + 0(\lambda^4 \log\lambda) \quad \text{as } \lambda \to 0 \tag{78a}$$

The integral representing ΔL_2 in Eq. (72) is indeterminate, but its principle value exists. The method of contour integration described previously in the section "General Formulation of the Problem" gives a complicated

expression. In this case we shall use the following approximation. By $\mu = (\kappa \sec^2\theta)u$ we can transform Eq. (72) into the following form:

$$\Delta L_2 = -\rho\Gamma_o^2 \int_0^\infty \frac{du}{u\sqrt{u-1}} \int_0^{\pi/2} e^{-\frac{2u}{\sigma}\sec^2\theta} \left[\frac{J_1(\frac{u}{\beta}\sec^2\theta \sin\theta)}{\sec^2\theta \sin\theta}\right]^2 \sec^4\theta \, d\theta \tag{79}$$

The inner integral is very similar to that representing D_4 in Eq. (63) which is discussed in detail in Appendix IV, D. This integral may be treated in a similar way (cf. App. IV, H). Because the interesting feature of ΔL_2 also comes from the integration with respect to u, we shall take only the first order term of the inner integral. The final result is as follows(cf. App. IV, H):

for σ very large (shallow submergence),

$$\Delta L_2 = \frac{\rho\Gamma_o^2}{4\sqrt{2}}\Gamma(\tfrac{1}{4})\left[1 - \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}\,\Gamma(\frac{1}{4})}\frac{\lambda}{\sqrt{1+\lambda^2}}\right]\left[1+\sqrt{\frac{2}{\pi\sigma}}(\gamma + \log\frac{2}{\sigma})\right](1+O(\lambda^2, \tfrac{1}{\sigma})) \; ; \tag{80}$$

and for σ very small,

$$\Delta L_2 \simeq \frac{\rho\Gamma_o^2}{8}\sqrt{\pi}\,(\tfrac{\sigma}{2})^{3/2}\frac{1}{\beta^2}\,(1 + \tfrac{\sigma}{2} + O(\sigma^2)) \,. \tag{81}$$

From these relations it is of interest to see that ΔL_2 tends to a constant value $\rho\Gamma_o^2\Gamma(\frac{1}{4})/4\sqrt{2}$ as h tends to zero. This result is quite a contrast to the feature of ΔL_1. Compared with ΔL_1, the value of ΔL_2 is smaller for σ large, and greater for σ small.

c. The Induced Velocity at the Hydrofoil

In order to investigate further the geometric and hydrodynamic properties of the hydrofoil, we need to know the value of Γ_o; and if we want to calculate Γ_o in terms of given quantities, we have first to obtain the values of the u-velocity and the downwash at the hydrofoil (cf. Eq. 41).

Substituting the value of $\Gamma(y)$ into Eq. (43), we may express $u(y)$ in terms of two parts :

$$u(y) = u_1(y) + u_2(y) \tag{82}$$

where

$$u_1(y) = -\frac{\Gamma_o}{2\pi b}\int_o^{\pi/2} d\theta \int_o^{\infty} e^{-2\lambda\mu}\,\frac{J_1(\mu\sin\theta)}{\sin\theta}\cos(\mu\eta\sin\theta)\,d\mu\,, \tag{82a}$$

$$u_2(y) = -\frac{\kappa\Gamma_o}{\pi}\int_o^{\pi/2} d\theta \int_o^{\infty} e^{-2\lambda\mu}\,\frac{J_1(\mu\sin\theta)}{\sin\theta}\cos(\mu\eta\sin\theta)\,\frac{\sec^2\theta}{\mu - b\kappa\sec^2\theta}\,d\mu \tag{82b}$$

and

$$\eta = \frac{y}{b}\,, \qquad \kappa = \frac{g}{U^2}\,, \qquad \lambda = \frac{h}{b}\,. \tag{82c}$$

The integral in Eq. (82a), which represents the effect due to the mean free surface, can be evaluated exactly. The result is as follows, (cf. App. IV, I) :

$$u_1(y) = -\frac{\Gamma_o}{8\sqrt{\pi}\,b}\,\frac{1}{\lambda\sqrt{1+4\lambda^2}}\sum_{n=o}^{\infty}\frac{(-)^n\Gamma(n+\frac{3}{2})}{n!}\,\frac{\eta^{2n}}{(1+4\lambda^2)^n}\,F\left(n+\frac{1}{2},\frac{1}{2}-n;2;\frac{1}{1+4\lambda^2}\right) \tag{83}$$

The value of $u_1(y)$ is always negative and becomes infinite as $h\to 0$. It can be shown that the above series converges uniformly with respect to η when $0\leq\eta\leq 1$. We may also find that $u_1(y)$ is a slowly varying function of y ; in particular, the value of $u_1(0)$ is

$$u_1(0) = -\frac{\Gamma_o}{16b}\,\frac{1}{\lambda\sqrt{1+4\lambda^2}}\;{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;\frac{1}{1+4\lambda^2}\right)$$

$$= -\frac{\Gamma_o}{4\pi b}\,\frac{1}{\lambda\sqrt{1+4\lambda^2}}\,B\left(\frac{1}{\sqrt{1+4\lambda^2}}\right) \tag{84}$$

where $B(k)$ is a derived complete elliptic integral defined by Eq. (74a).

It can be shown that $u_2(y)$ is also a slowly varying function of y, i.e. the values of $u_2(y)$ at points close to the hydrofoil differ very slightly from its value at plane of symmetry given by (cf. App. IV, J) :

$$u_2(0) \cong -\frac{2\Gamma_o}{\pi b}\,\frac{1}{\sqrt{2\lambda}}\,\frac{1}{(1+4\lambda^2)^{1/4}}\,\frac{1}{\sqrt{1+4\lambda^2}+2\lambda}\,B\left(\sqrt{\frac{1}{2}-\frac{\lambda}{\sqrt{1+4\lambda^2}}}\right)\left[1+\sqrt{\frac{2}{\pi\sigma}}\left(\gamma+\log\frac{2}{\sigma}\right)\right]\left(1+O\left(\frac{1}{\sigma}\right)\right)$$

for σ large ; (85a)

and

$$u_2(0) \cong \frac{\Gamma_o}{8b} \frac{1}{\sqrt{2\pi\sigma}} \cdot \frac{1}{\lambda^2} (1 + 0(\sigma)), \quad \text{for } \sigma \text{ small} \tag{85b}$$

It is of interest to see that $u_1(0)$ and $u_2(0)$ are of the same order of magnitude. Therefore the total u-component velocity at $y = 0$ is

$$u(0) \cong - \frac{\Gamma_o}{\pi b} \left\{ \frac{1}{4\lambda\sqrt{1+4\lambda^2}} B\left(\frac{1}{\sqrt{1+4\lambda^2}}\right) + \frac{2}{\sqrt{2\lambda}(1+4\lambda^2)^{1/4}\left[\sqrt{1+4\lambda^2}+2\lambda\right]} B\left(\sqrt{\frac{1}{2} - \frac{\lambda}{\sqrt{1+4\lambda^2}}}\right)\left[1 + \right.\right.$$

$$\left.\left. + \sqrt{\frac{2}{\pi\sigma}}\left(\gamma + \log\frac{2}{\sigma}\right)\right] \left(1 + 0\left(\frac{1}{\sigma}\right)\right)\right\}, \quad \text{for } \sigma \text{ large}; \tag{86a}$$

and

$$u(0) \cong - \frac{\Gamma_o}{8\pi b} \frac{1}{\lambda^2} \left(1 - \sqrt{\frac{\pi}{2\sigma}}\right)\left(1 + 0\left(\frac{1}{\lambda^2}, \sigma\right)\right), \quad \text{for } \sigma \text{ small.} \tag{86b}$$

The downwash at the hydrofoil can be calculated in a similar way:

$$w(y) = -\varphi_z(0, y, -h) = -(\varphi_{1_z} + \varphi_{2_z} + \varphi_{3_z} + \varphi_{4_z})_{(0,\, y,\, -h)} \tag{87a}$$

where

$$\varphi_{1_z}(0, y, -h) = -\frac{\Gamma_o}{4}\int_0^\infty J_1(\mu b)\cos(\mu y)\, d\mu$$

$$= \begin{cases} -\dfrac{\Gamma_o}{4b} & |y| \leq b \\[2ex] -\dfrac{\Gamma_o}{4b}\left(1 - \dfrac{|y|}{\sqrt{y^2 - b^2}}\right) & |y| > b \end{cases}, \tag{87b}$$

$$(\varphi_{2_z} + \varphi_{3_z})_{(0,\, y,\, -h)} = \frac{\Gamma_o}{4}\int_0^\infty e^{-2h\mu} J_1(b\mu)\cos\mu y \, d\mu, \tag{87c}$$

and

$$\varphi_{4_z}(0, y, -h) = -\kappa\Gamma_0 \int_0^{\pi/2} e^{-2h\kappa \sec^2\theta} \frac{J_1(\kappa b \sec^2\theta \sin\theta)}{\sec^2\theta \sin\theta} \cos(\kappa y \sec^2\theta \sin\theta) \sec^5\theta \, d\theta \tag{87d}$$

Now the integral in Eq. (87c) can be evaluated while that in Eq. (87d) again has to be approximated.

$$\frac{\partial}{\partial z}(\varphi_2 + \varphi_3)_{(0, y, -h)} = \frac{\Gamma_0}{4b}\left\{1 - R\ell \frac{2h - i|y|}{\sqrt{(2h - i|y|)^2 + b^2}}\right\} \quad \text{(Ref. 25, p. 33)}$$

$$= \frac{\Gamma_0}{4b}\left\{1 - \frac{1}{\sqrt{2}} F_1(\lambda, \eta)\right\} \quad . \tag{88a}$$

where

$$F_1(\lambda, \eta) = \frac{2\lambda \sqrt{\left[(4\lambda^2 + 1 - \eta^2)^2 + (4\lambda\eta)^2\right]^{\frac{1}{2}} + (4\lambda^2 + 1 - \eta^2)}}{\sqrt{(4\lambda^2 + 1 - \eta^2)^2 + (4\lambda\eta)^2}}$$

$$+ \frac{|\eta| \sqrt{\left[(4\lambda^2 + 1 - \eta^2)^2 + (4\lambda\eta)^2\right]^{\frac{1}{2}} - (4\lambda^2 + 1 - \eta^2)}}{\sqrt{(4\lambda^2 + 1 - \eta^2)^2 + (4\lambda\eta)^2}} \tag{88b}$$

and

$$\lambda = \frac{h}{b} , \qquad \eta = \frac{y}{b} \quad . \tag{88c}$$

It can be seen that $F_1(\lambda, \eta)$ is a slowly varying function of η for fixed λ. In particular, we have

$$F_1(\lambda, 0) = \frac{2\sqrt{2}\,\lambda}{\sqrt{1 + 4\lambda^2}} \quad , \tag{89a}$$

$$F_1(\lambda, \pm 1) = \frac{\lambda\sqrt{\lambda\sqrt{1+\lambda^2} + \lambda^2} + \sqrt{\lambda\sqrt{1+\lambda^2} - \lambda^2}}{\lambda\sqrt{1 + \lambda^2}} \tag{89b}$$

For small values of σ, application of the method of steepest descent to the integral in Eq. (87d) yields

$$\varphi_{4_z}(0, y, -h) \cong -\frac{\Gamma_o}{2b}\sqrt{\frac{\pi\sigma}{2}}\; e^{-2/\sigma}\,(1+0(\sigma)) \tag{90}$$

which diminishes exponentially with increasing h and is independent of y for points close to the hydrofoil. For very large σ, the integral in Eq. (87d) also represents a slowly varying function of y and consequently only the value of $\varphi_{4_z}(0, 0, -h)$ is of interest. It can be shown that (cf. App. IV, K):

$$\varphi_{4_z}(0, 0, -h) \cong -\frac{\Gamma_o}{2b}\left\{1 - \frac{2\lambda}{\sqrt{1+4\lambda^2}}\right\}\left[1 + 0(\frac{1}{\sigma}, \frac{1}{\beta})\right]$$

$$\cong -2\frac{\partial}{\partial z}(\varphi_2 + \varphi_3)_{(0,0,-h)}\left[1 + 0(\frac{1}{\sigma}, \frac{1}{\beta})\right]. \tag{91}$$

Finally we have the value of $w(0)$, by substituting Eqs. (87a), (88)-(91) into Eq. (87a), as follows

$$w(0) \cong +\frac{\Gamma_o}{2b}\left\{1 - \frac{\lambda}{\sqrt{1+4\lambda^2}}\right\}\left[1 + 0(\frac{1}{\sigma}, \frac{1}{\beta})\right], \quad \text{for } \sigma \text{ large;} \tag{92a}$$

$$w(0) = +\frac{\Gamma_o}{4b}\left\{1 + \frac{1}{8\lambda^2}\right\}\left[1 + 0(\frac{1}{\lambda^4}, \sqrt{\sigma}\, e^{-2/\sigma})\right], \quad \text{for } \sigma \text{ small,} \tag{92b}$$

which tends to aerodynamic value

$$w_\infty(0) = +\frac{\Gamma_\infty(0)}{4b} \tag{92c}$$

as $h \to \infty$.

It may be remarked here that both $u(y)$ and $w(y)$ are almost constant spanwise at points close to the hydrofoil. This fact justifies some of the basic assumptions introduced previously in the sections, "General Formulation of the Problem" and "Geometry of the Hydrofoil; Effective Angle of Attack". This result also supports the assumption that the same circulation distribution will hold for a wide range of depths.

d. Geometry of the Hydrofoil

We are interested primarily in the case of shallow immersion, because the geometry of the hydrofoil at deep immersion will be the same as the

corresponding aerodynamic problem. Substituting Eqs. (86) and (92) into (41a) and neglecting second order small quantities, we obtain the value of Γ_o ($\equiv \Gamma(0)$) as a function of λ and σ as follows:

$$\Gamma_o = \frac{\frac{1}{2} U c_o a_e a_a}{1 + \frac{1}{4}\frac{c_o}{b} a_e \left[f_2(\lambda, \sigma) + \frac{2}{\pi} a_a f_1(\lambda, \sigma) \right]} \tag{93a}$$

where

$$f_1(\lambda, \sigma) = \frac{1}{4\lambda \sqrt{1+4\lambda^2}} B\left(\frac{1}{\sqrt{1+4\lambda^2}}\right) + \frac{2B\left(\sqrt{\frac{1}{2} - \frac{\lambda}{\sqrt{1+4\lambda^2}}}\right)}{\sqrt{2\lambda}\,(1+4\lambda^2)^{1/4}\left[\sqrt{1+4\lambda^2}+2\lambda\right]} \left[1 + \sqrt{\frac{2}{\pi\sigma}}\left(\gamma + \log \frac{2}{\sigma}\right)\right] + 0\left(\frac{1}{\sigma}\right) \tag{93b}$$

and

$$f_2(\lambda, \sigma) = \left[1 - \frac{\lambda}{\sqrt{1+4\lambda^2}}\right]\left[1 + 0\left(\frac{1}{\sigma}, \frac{1}{\beta}\right)\right], \tag{93c}$$

As h tends to ∞, f_1 tends to zero and f_2 tends to $\frac{1}{2}$; hence the value of Γ_o at infinite depth becomes

$$\Gamma_{o_\infty} = \frac{\frac{1}{2} U c_o a_e a_a}{1 + \frac{1}{8}\frac{c_o}{b} a_e} \tag{94}$$

which is the well-known aerodynamic value of Γ in terms of given quantities. It is of interest to note that Γ_o changes with respect to $\Gamma_\infty(0)$, as λ and σ vary, according to the following relation:

$$\frac{\Gamma_o}{\Gamma_{o_\infty}} = \frac{1 + \frac{1}{8}\frac{c_o}{b} a_e}{1 + \frac{1}{4}\frac{c_o}{b} a_e \left[f_2(\lambda, \sigma) + \frac{2}{\pi} a_a f_1(\lambda, \sigma) \right]} = \gamma(\lambda, \sigma, a_a) \tag{95}$$

which is a function of λ, σ and a_a for a given hydrofoil. This function and its squared value, γ^2, are plotted against λ in Fig. 3 for two incidences $a_a = 3^o$ and 6^o under the following operating condition:

$$c_o = 8 \text{ ft}, \quad b = 20 \text{ ft}, \quad a_e \doteq 2\pi, \quad U = 100 \text{ ft/sec}, \quad \sigma = \frac{U^2}{gh} = \frac{15.5}{\lambda} \tag{96a}$$

so that

$$\gamma(\lambda, \frac{15.5}{\lambda}, a_a) = \frac{\Gamma_o}{\Gamma_{o_\infty}} = \frac{1.314}{1 + 0.628\left[f_2(\lambda, \frac{15.5}{\lambda}) + \frac{2}{\pi} a_a f_1(\lambda, \frac{15.5}{\lambda})\right]} \tag{96b}$$

The values of γ and γ^2 decrease with decreasing λ, and tend to unity asymptotically as $\lambda \to \infty$. For small values of λ ($\lambda < 1$), both Γ_o and Γ_o^2 deviate appreciably from their respective values at infinite depth.

It may be pointed out here that because both $u(y)$ and $w(y)$ are almost constant along the span of the hydrofoil, it follows from Eq. (41) that for an elliptical distribution of $\Gamma(y)$, the plan form is also nearly elliptical,

$$c(y) = c_o \sqrt{1 - \frac{y^2}{b^2}}, \tag{97}$$

together with a negligible geometric twist.

Another consequence of the above results is that the condition of Eq. (55) for minimum drag with prescribed lift is also approximately satisfied by almost constant (spanwise) $u(y)$ and $w(y)$. Hence the hydrofoil of elliptical plan form still gives approximately the minimum drag for a given lift.

e. Over-all Hydrodynamic Properties; Lift and Drag Coefficient

In this section we shall again only consider the case of shallow submergence. Combining our previous results we obtain the total lift and drag of the hydrofoil at small depths as follows:

$$L \doteq \frac{\pi}{2} \rho U b \Gamma_{o_\infty} \gamma - \frac{\rho \Gamma_{o_\infty} \gamma^2}{3\pi} \left[f_3(\lambda) - f_4(\lambda, \sigma)\right] \tag{98a}$$

where

$$f_3(\lambda) = \frac{1}{\lambda\sqrt{1+\lambda^2}} \left\{ B\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - \frac{\lambda^2}{\sqrt{1+\lambda^2}} C\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - \frac{3}{8}\lambda^2 \log \frac{\sqrt{1+\lambda^2}}{\lambda} \right\}, \tag{98b}$$

$$f_4(\lambda, \sigma) = \frac{3\pi}{4\sqrt{2}} \Gamma(\tfrac{1}{4}) \left(1 - \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}\,\Gamma(\frac{1}{4})} \frac{\lambda}{\sqrt{1+\lambda^2}}\right) \left[1 + \sqrt{\frac{2}{\pi\sigma}} \left(\gamma + \log\frac{2}{\sigma}\right)\right]$$

$$= 6.05 \left(1 - 0.19 \frac{\lambda}{\sqrt{1+\lambda^2}}\right) \left[1 + \sqrt{\frac{2}{\pi\sigma}} \left(\gamma + \log\frac{2}{\sigma}\right)\right] \tag{98c}$$

and Γ_{o_∞} and $\gamma(\lambda, \sigma, a_a)$ are given by Eqs. (94) and (95) respectively.

$$D \doteq \frac{\pi \rho \Gamma_{o_\infty}^{2}}{8} \gamma^2 \left\{ f_5(\lambda) + 2\left[1 - f_5(\lambda)\right] \right\} , \tag{99a}$$

where

$$f_5(\lambda) = \frac{4}{\pi} \lambda \sqrt{1+\lambda^2} \left[K\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - E\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \right] . \tag{99b}$$

At this stage we should be able to see the properties of the total drag exerted on the hydrofoil. As mentioned before, when $\lambda \to \infty$ D tends to the aerodynamic value $D_\infty = \pi \rho \Gamma_{o_\infty}^{2}/8$, hence we write

$$\frac{D}{D_\infty} = \gamma^2 f_5(\lambda) + 2\gamma^2 \left[1 - f_5(\lambda)\right] . \tag{99c}$$

This ratio is plotted in Fig. 4 for $a_a = 6^\circ$ and under the operating condition given by Eq. (96a). The first term on the right hand side represents the contribution from the total induced drag; this quantity tends to D_∞ asymptotically for large h and decreases rapidly as the hydrofoil approaches the surface ($h < 2b$). The second term represents approximately the wave drag which is negligibly small for $h > 2b$ and becomes important only at small depths. As a whole, the total drag D deviates only slightly from D_∞ for $h > \frac{b}{2}$, and decreases moderately at small depths ($h < \frac{1}{2} b$) which is actually the range of interest. When the hydrofoil approaches the planning condition, the total drag is solely due to the wave effect.

In order to measure the total drag, including the wave drag, exerted on a hydrofoil, the experiment must be carried out in a towing tank. Then it would be convenient to define the over-all lift and drag coefficient with respect to the towing speed, such as,

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 S} , \qquad C_D = \frac{D}{\frac{1}{2}\rho U^2 S} \tag{100}$$

where ρ is the density of the liquid, U the towing speed, and S the area of the plan form. The above definition for C_L and C_D were used in previous experiments (known to the author). In order to compare with these available experimental results, we shall also adopt the above definition of C_L and C_D. Substituting Eqs. (94), (98) and (99) into (100), we obtain

$$C_L \doteq \frac{\pi}{8} \mathcal{R} \left(\frac{c_o}{b}\right) \frac{a_e a_a \gamma}{\left(1 + \frac{1}{8}\frac{c_o}{b} a_e\right)} - \frac{1}{24\pi} \mathcal{R} \left(\frac{c_o}{b}\right)^2 \left[\frac{a_e a_a \gamma}{1 + \frac{1}{8}\frac{c_o}{b} a_e}\right]^2 \left\{f_3(\lambda) - f_4(\lambda, \sigma)\right\}, \tag{101}$$

$$C_D = \frac{\pi}{64} \mathcal{R} \left(\frac{c_o}{b}\right)^2 \left[\frac{a_e a_a \gamma}{1 + \frac{1}{8}\frac{c_o}{b} a_e}\right]^2 \left\{f_5(\lambda) + 2\left[1 - f_5(\lambda)\right]\right\}, \tag{102}$$

where $\mathcal{R} = \frac{4b^2}{S}$ = aspect ratio and f_3, f_4, f_5 are given in Eqs. (98) and (99). The lift drag ratio is

$$\frac{C_L}{C_D} \doteq \frac{1 - \frac{1}{3\pi^2}\left(\frac{c_o}{b}\right)\left[\frac{a_e a_a \gamma}{1 + \frac{1}{8}\frac{c_o}{b} a_e}\right]\left\{f_3(\lambda) - f_4(\lambda, \sigma)\right\}}{\frac{1}{8}\left(\frac{c_o}{b}\right)\left[\frac{a_e a_a \gamma}{1 + \frac{1}{8}\frac{c_o}{b} a_e}\right]\left\{f_5(\lambda) + 2\left[1 - f_5(\lambda)\right]\right\}}. \tag{103}$$

These results should be good for $\lambda < 2$ $(h < 2b)$ provided that $\sigma = \frac{U^2}{gh}$ is large, say, greater than 20. From the expressions of L and D for h large, we note that as $h \to \infty$, both C_L and C_D tend asymptotically to their aerodynamic values, namely,

$$C_L \cong \frac{\pi}{8} \mathcal{R} \left(\frac{c_o}{b}\right) \frac{a_e a_a}{1 + \frac{1}{8}\frac{c_o}{b} a_e}, \quad C_D \cong \frac{\pi}{64} \mathcal{R} \left(\frac{c_o}{b}\right)^2 \left[\frac{a_e a_a}{1 + \frac{1}{8}\frac{c_o}{b} a_e}\right]^2.$$

so that

$$C_D \cong \frac{1}{\pi \mathcal{R}} C_L^2$$

which agrees with aerodynamic wing theory. However, it can also be seen from Eqs. (101) and (102) that even for λ small, C_D/C_L^2 is still proportional to $1/\mathcal{R}$ if the small quantity of the second order is neglected. From this result it follows that in order to improve the hydrodynamic properties the preference should be for a high aspect ratio hydrofoil, even for operations near the water surface.

To illustrate the details of the behavior, and to compare our results with experiments, we shall take specific values of c_o, b, $\mathcal{R}$, and a_e to determine all the coefficients. This will be shown as follows.

DISCUSSION OF RESULTS

Consider a hydrofoil of elliptical plan form with other specifications and an operating condition given by Eq. (96a) so that $AR = 6.3$. These data serve to determine the coefficients in Eqs. (101) - (103) which then become

$$C_L \doteq 4.74\, a_a \gamma - 0.306\, a_a^2 \gamma^2 \left\{ f_3(\lambda) - f_4(\lambda,\sigma) \right\} , \tag{104}$$

$$C_D \doteq 1.14\, a_a^2 \gamma^2 \left\{ f_5(\lambda) + 2\left[1 - f_5(\lambda)\right] \right\} \tag{105}$$

and

$$\frac{C_L}{C_D} \doteq \frac{1 - 0.065\, a_a \gamma \left\{ f_3(\lambda) - f_4(\lambda,\sigma) \right\}}{0.24\, a_a \gamma \left\{ f_5(\lambda) + 2\left[1 - f_5(\lambda)\right] \right\}} . \tag{106}$$

These equations are plotted against λ for $a_a = 6^\circ$ in Fig. 5 and also plotted against a_a for several small values of λ in Fig. 6. From these curves several interesting conclusions may be drawn.

At depths greater than 2 chords, the influence of the surface of the water is negligibly small and the hydrofoil will have characteristics similar to those of an airfoil of the same section. In the range of depths less than 2 chords, C_D decreases gradually because the decrease of liquid flow (with respect to U) above the foil diminishes the induced downwash. At very small depths $(h < c_o/2)$, the wave drag only is important; our results show that when the hydrofoil is near the surface, the rate of energy shed to form the wave system is slightly less than the induced drag at large depths. In the same range of operation $(h < 2c_o)$, C_L decreases comparatively rapidly to almost $\frac{1}{2} C_{L\infty}$ as the hydrofoil approaches the surface. This reduction in lift results from the decrease of mass of water flowing over the upper surface of the hydrofoil, causing a reduction of the absolute value of the negative pressure on the suction side. The corresponding lift-drag ratio decreases very slowly with decrease in depths for $h > c_o/2$, and is almost constant about the point $h = \frac{1}{2} c_o$. This ratio decreases rapidly with further decrease in depth for $h < \frac{1}{2} c_o$. This result also indicates that if the water surface is not too choppy, then the depths between $\frac{1}{2} c_o$ and $1\, c_o$ would be a favorable range for operation. This optimum range of depths also associates with a stabilizing effect because this range corresponds to the middle part of increasing slope $dC_L/d\lambda$, consequently, any further

decrease in depths will result in a rapid decrease of lift so the hydrofoil will sink, and if it sinks, an increase in lift will raise it up again. The above results as shown in Fig. 5 agree well with some of previous experiments (cf. Ref. 17, 18, 4).

If we add an almost constant frictional drag (about $C_{D_f} = 0.02$) to the drag coefficient, then the result shown in Fig. 6 is in good agreement with the observations made in Ref. 21 and 18. The reduction in C_L and C_D with decrease in depths becomes more appreciable at larger angles of attack.

It is also of interest to consider the situation of practical operation. When a hydrofoil moves through water of infinite surface extent at shallow submergence, an observer who moves with the hydrofoil can measure the value of the approaching flow velocity $U + u(0)$ more easily (for instance, with a pitot tube) than the value of U at upstream infinity. Then if one defines the overall lift and drag coefficient based on the value of $U + u(0)$, that is

$$C_L' = \frac{L}{\frac{1}{2}\rho U^2 (1 + \frac{u(0)}{U})^2 S}, \qquad C_D' = \frac{D}{\frac{1}{2}\rho U^2 (1 + \frac{u(0)}{U})^2 S} \tag{107}$$

where

$$\frac{u(0)}{U} = -\frac{1}{2\pi} \frac{(\frac{c_o}{b}) a_e \alpha_a}{1 + \frac{1}{8}\frac{c_o}{b} a_e} \left[\gamma \cdot f_1(\lambda, \sigma)\right], \tag{108}$$

one will find that both C_L' and C_D' are almost constant for $\lambda > 0.1$ (cf. Fig. 7) although C_L'/C_D' is identical to C_L/C_D. However, the total lift L and the total drag D calculated from Eq. (107) would still have the same dependence on λ as C_L and C_D, only with a different proportionality constant.

The above results are derived under the assumption of large Froude numbers, the effect of speed U on C_L and C_D is dropped out from our final formulas. This effect becomes significant for motions with small Froude number (say, less than 1). For instance, let us consider the same hydrofoil as that in the previous case, but moving at a low speed,

$$U = 10 \text{ ft/sec}, \quad c_o = 8 \text{ ft}, \quad b = 20 \text{ ft}, \quad a_e \doteq 2\pi \tag{109a}$$

then

$$\beta = \frac{U^2}{bg} = 0.156 \ , \qquad \sigma = \frac{0.156}{\lambda}$$

which may be considered as small for $\lambda > \frac{1}{2}$. Using the values of $u(0)$ and $w(0)$ for σ small (cf. Eqs. 86, 92 and 41a), we obtain the value of $\gamma(\lambda, \sigma, a_a)$ in this range of operation to be

$$\gamma(\lambda, \sigma, a_a) = \frac{\Gamma_o}{\Gamma_{o\infty}} = \frac{1 + \frac{1}{8}\frac{c_o}{b} a_e}{1 + \frac{1}{8}\frac{c_o}{b} a_e \left[g_2(\lambda) + \frac{a_a}{2\pi} g_1(\lambda) \right]} \qquad (\sigma < \tfrac{1}{3}) \tag{110a}$$

where

$$g_1(\lambda) \doteq \frac{1}{\lambda^2} \left[1 - \sqrt{\frac{\pi \lambda}{2\beta}} \right] \tag{110b}$$

$$g_2(\lambda) \doteq 2 \left[1 - \frac{\lambda}{\sqrt{1 + 4\lambda^2}} \right] \tag{110c}$$

The value of γ given by Eq. (110) is plotted in Fig. 8 for $a_a = 6^o$, using the specification given by Eq. (109). The result shows that γ also decreases with decreasing λ for σ small, but the rate of decrease is slower than that in large σ case.

Since the Froude number effect only enters in the expression of D_4 and ΔL_2, it suffices to discuss only the behavior of these two quantities. With respect to a convenient reference, say, D_∞, we have the values of D_4 and ΔL_2 for σ small as follows:

$$\frac{D_4}{D_\infty} \doteq \gamma^2 \sqrt{\frac{\pi}{2\lambda}} \ \beta^{-\frac{3}{2}} \ e^{-\frac{2\lambda}{\beta}} \tag{111}$$

$$\frac{\Delta L_2}{D_\infty} \doteq \gamma^2 \frac{1}{\sqrt{2\pi\beta}} \ \lambda^{-\frac{3}{2}} \tag{112}$$

Noting that β is proportional to U^2, it can be seen from above relations that for small Froude numbers ($\sim$ low speeds) an increase in speed will cause both C_L and C_D to decrease, with the drag decreasing more

rapidly. This effect has also been found experimentally at small values of U (cf. Refs. 17, 21). Eqs. (111) and (112) are plotted in Fig. 9 with the specification given by (109). In the present range of operation ($\sigma = \frac{.156}{\lambda}$, $\lambda > \frac{1}{2}$) the wave drag is negligibly small compared with D_∞, but ΔL_2 increases very rapidly, relative to D_∞, with decrease in depth. The total change in ΔL_2 in this range is, however, still quite small with respect to L_o.

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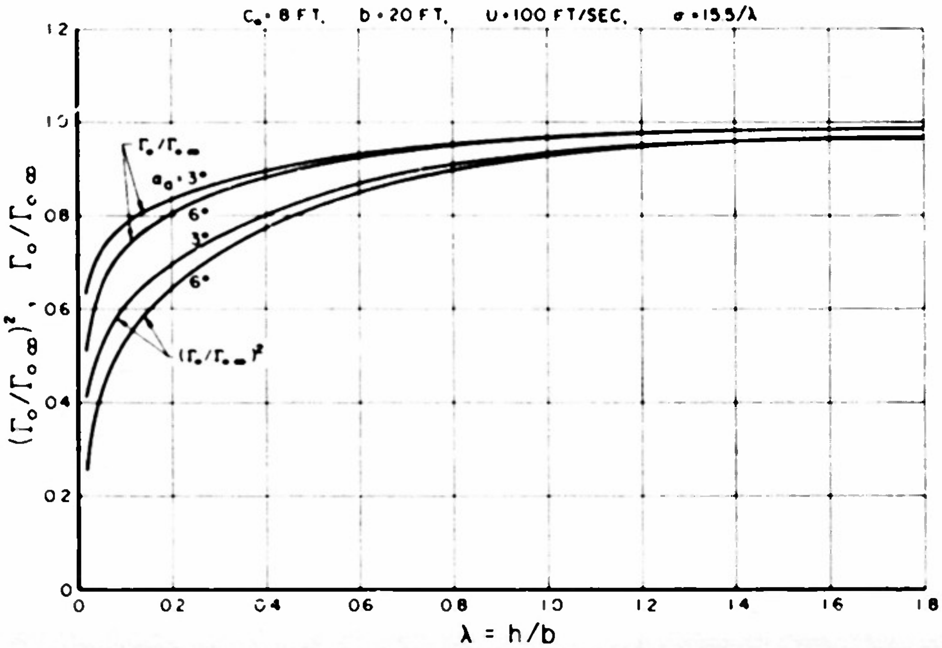


Fig. 3 - Effect of depth on circulation strength.

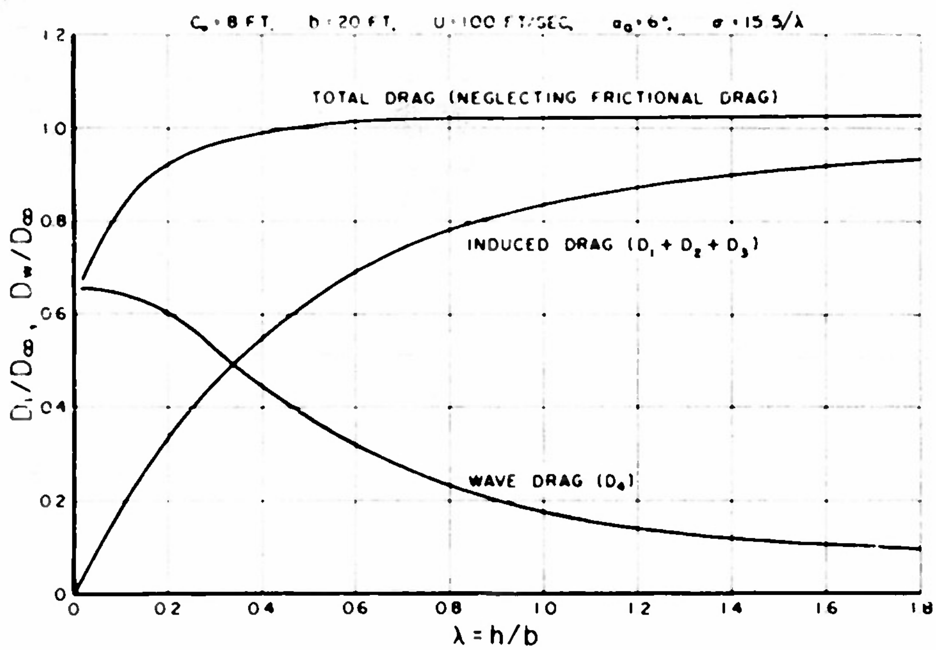


Fig. 4 - Effect of depth on drag.

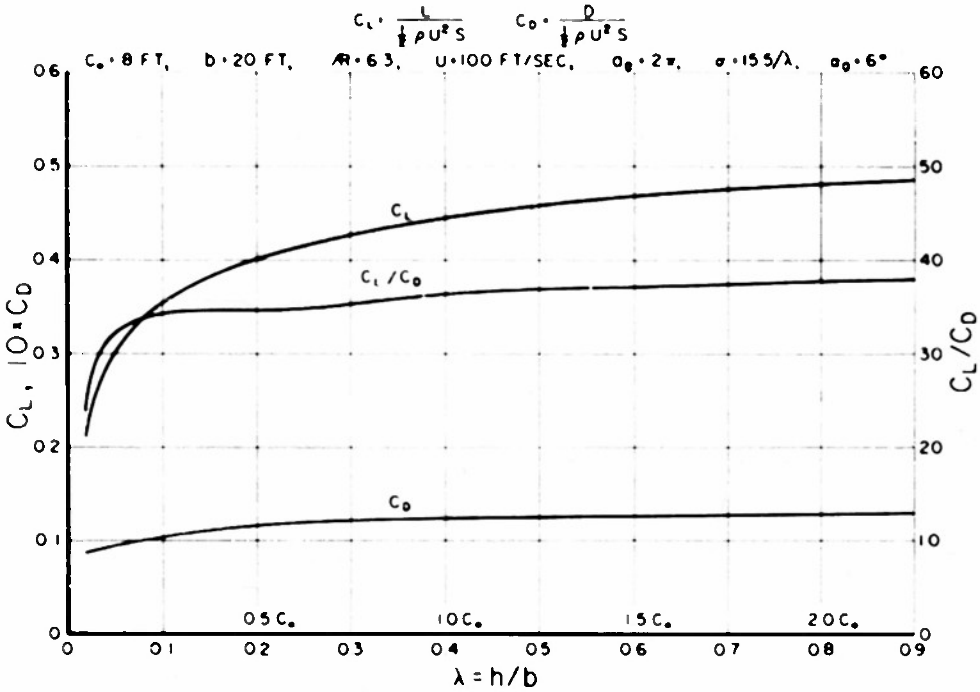


Fig. 5 - Effect of depth on C_L, C_D and C_L/C_D.

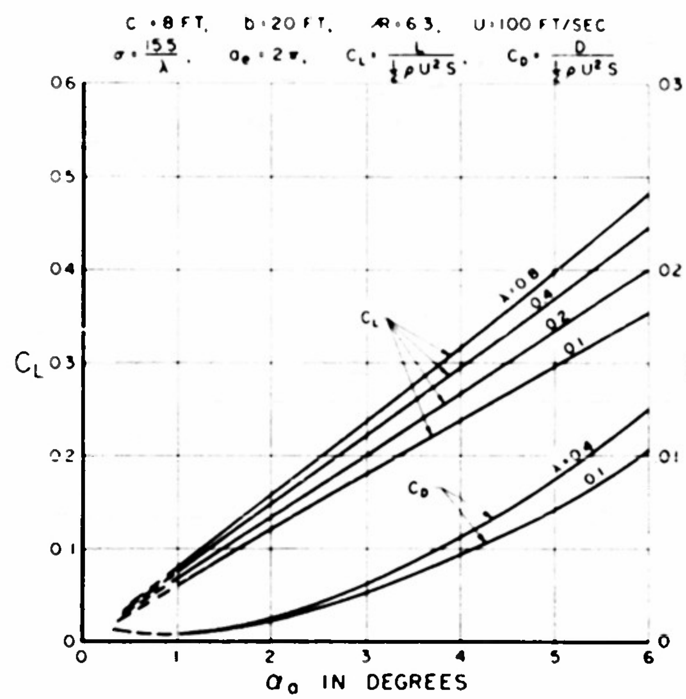


Fig. 6 - Hydrodynamic characteristics of the hydrofoil at shallow submergence.

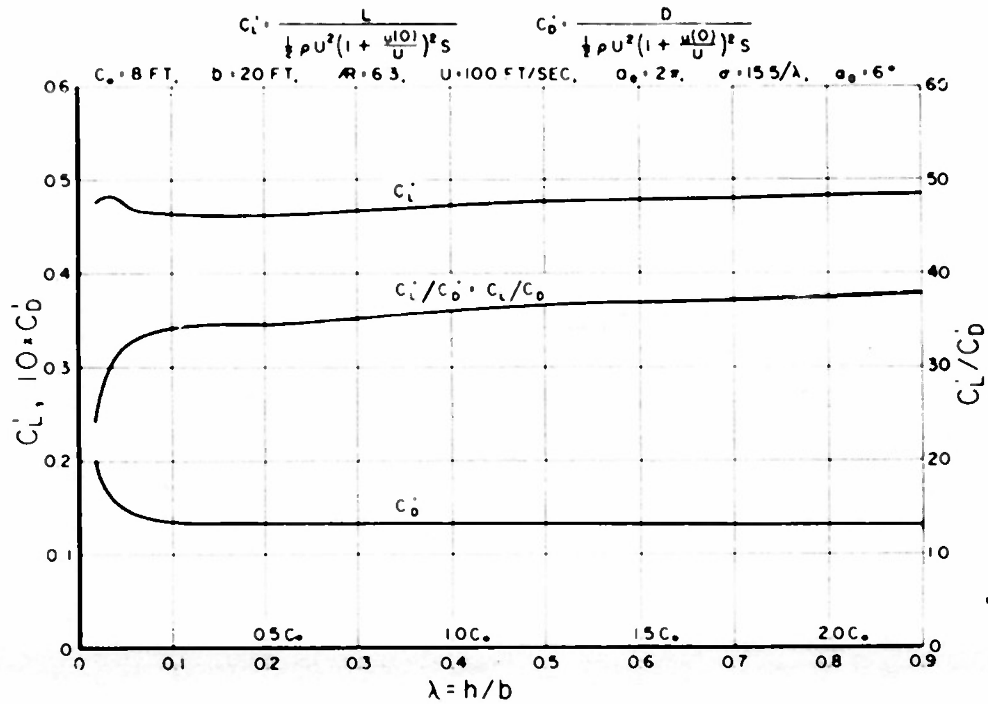


Fig. 7 - Effect of depth on C_L', C_D', (based on approaching flow velocity).

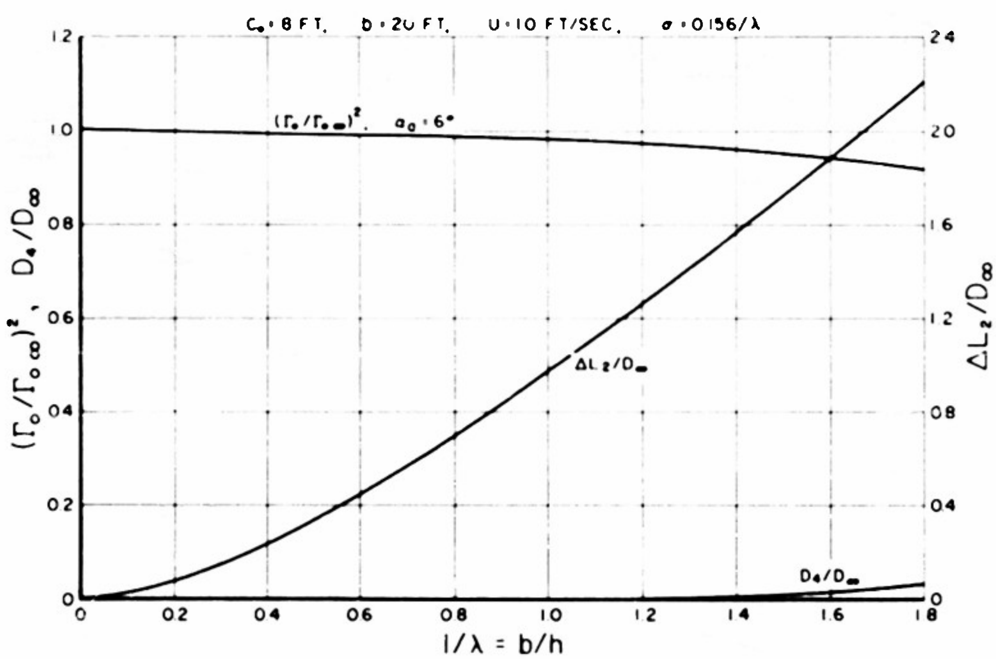


Fig. 8 - Effect of depth at small values of Froude number.

APPENDIX I

Some Integral Representations of a Lifting Line

It is given that a lifting line of span $2b$ is located along y-axis from $-b$ to b with a known distribution of circulation $\Gamma(y)$, (see Fig. 9), the free stream being uniform, of velocity U in x-direction. The problem is to find the induced velocity potential $\varphi(x, y, z)$ due to this lifting line.

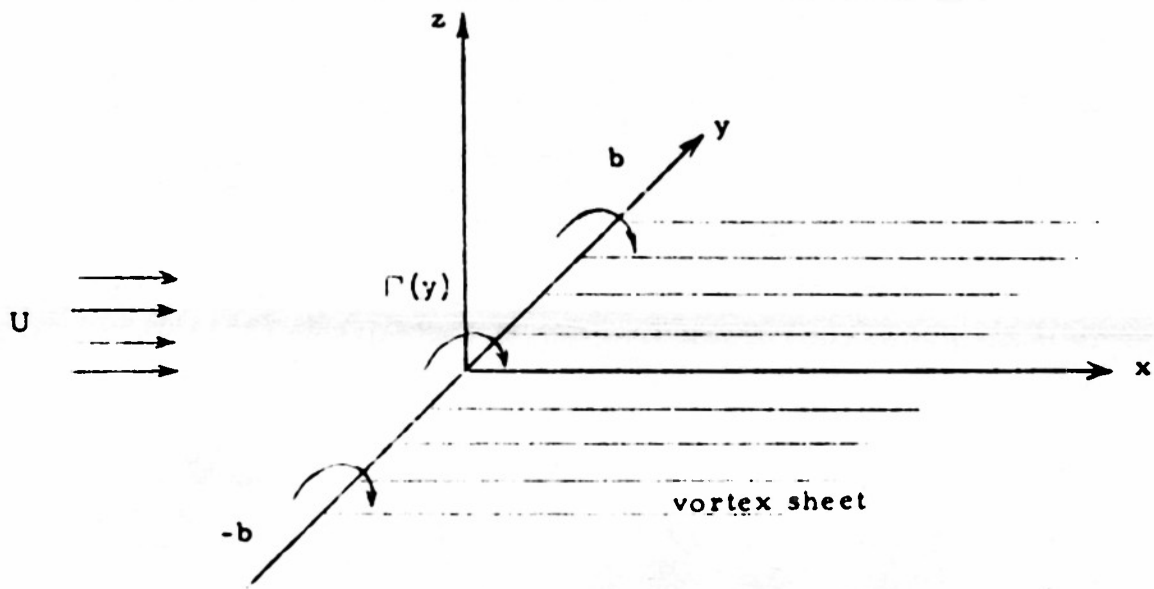


Fig. 9 - A lifting line with its trailing vortex sheet.

First this potential φ should satisfy the Laplace equation

$$\nabla^2\varphi = 0 , \tag{I. 1}$$

Next we shall impose boundary conditions for this problem. According to the lifting line theory, the approximation may be made that the trailing vortex sheet is parallel to the free stream; that is, on the surface $z = 0$, with $x > 0$, $|y| \leq b$. It is easy to see that boundary conditions for φ should be as follows (see, for example, Ref. 22) :

(i) $\varphi = 0$ on $z = 0$, outside the vortex sheet;

(ii) Across the vortex sheet, φ has a jump $\Gamma(y)$

$$(x, y, 0\pm) = \pm \frac{\Gamma(y)}{2} \quad \text{on the vortex sheet;} \tag{I.2}$$

(iii) $\frac{\partial\varphi}{\partial z}$, $\frac{\partial\varphi}{\partial x}$ are continuous and $\frac{\partial\varphi}{\partial y}$ discontinuous across the vortex sheet, but

$$\left|\frac{\partial\varphi}{\partial y}\right|_{z=0+} = \left|\frac{\partial\varphi}{\partial y}\right|_{z=0-}, \text{ so that the pressure is continuous there.}$$

(iv) φ and grad $\varphi \to 0$ as $\sqrt{y^2 + z^2} \to \pm\infty$ and/or $x \to -\infty$.

Conditions (i) - (iv) also imply that

(v) $\varphi(x, y, z) = -\varphi(x, y, -z)$;

(vi) $\frac{\partial}{\partial x}\varphi(x, y, z) = -\frac{\partial}{\partial x}(x, y, -z)$;

(vii) $\varphi(x, y, \pm 0) = \pm\frac{\Gamma(y)}{4}(1 + \text{sign } x)$.

One way to solve (I.1) together with the boundary conditions (I.2) is by a Fourier transformation. Define the double Fourier transformation of $\varphi(x, y, z)$ with respect to x and y by

$$\tilde{\varphi}(\lambda, \mu, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\, d\lambda \int_{-\infty}^{\infty} e^{-i\mu y}\, \varphi(x, y, z)\, dy \tag{I.3}$$

so that the inversion formula is given by

$$\varphi(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\, d\lambda \int_{-\infty}^{\infty} e^{i\mu y}\, \tilde{\varphi}(\lambda, \mu, z)\, d\mu \tag{I.4}$$

Applying (I.3) to (I.1), we have

$$\left(\frac{d^2}{dz^2} - \lambda^2 - \mu^2\right)\tilde{\varphi} = 0 .$$

The solution which satisfies (iv) and (v) has the following form

$$\tilde{\varphi}(\lambda, \mu, z) = (\text{sign } z)\, A(\lambda, \mu)\, e^{-\sqrt{\lambda^2+\mu^2}\,|z|}\,, \tag{I. 5}$$

so that

$$\tilde{\varphi}(\lambda, \mu, 0+) = A(\lambda, \mu)\,. \tag{I. 6}$$

Application of the same transformation (I. 3) to (vii) gives:

$$\tilde{\varphi}(\lambda, \mu, 0+) = \frac{1}{8\pi}\int_{-\infty}^{\infty} (\eta) e^{-i\mu\eta}\, d\eta \left[\int_{-\infty}^{\infty} e^{-i\lambda x} \text{ sign } x\, dx + \int_{-\infty}^{\infty} e^{-i\lambda x}\, dx\right]\,.$$

Using the conventional method of summability, we have

$$\int_{-\infty}^{\infty} e^{-i\lambda x} \text{ sign } x\, dx = \frac{2}{i} \lim_{\epsilon \to 0+} \int_{0}^{1/\epsilon} e^{-x} \sin \lambda x\, dx = \frac{2}{i\lambda}$$

and

$$\int_{-\infty}^{\infty} e^{-i\lambda x}\, dx = 2\pi\delta(\lambda)\,.$$

Hence we also have

$$\tilde{\varphi}(\lambda, \mu, 0+) = \frac{1}{4\pi}\left[\frac{1}{i\lambda} + \pi\,\delta(\lambda)\right]\int_{-\infty}^{\infty} \Gamma(\eta) e^{-i\mu\eta}\, d\eta \tag{I. 7}$$

Comparison of (I. 7) with (I. 6) gives

$$A(\lambda, \mu) = \frac{1}{4\pi}\left[\frac{1}{i\lambda} + \pi\,\delta(\lambda)\right]\int_{-\infty}^{\infty} \Gamma(\eta)\, e^{-i\mu\eta}\, d\eta \tag{I. 8}$$

Substituting (I. 8) into (I. 5) and then applying the inversion formula (I. 4), we obtain the solution

$$\varphi = \text{sign } z\left[\frac{1}{2\pi^2}\int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_{0}^{\infty} \frac{\sin \lambda x}{\lambda}\, d\lambda \int_{0}^{\infty} \cos \mu(y-\eta)\, e^{-\sqrt{\lambda^2+\mu^2}\,|z|}\, d\mu\right.$$

$$\left. + \frac{1}{4\pi}\int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_{0}^{\infty} \cos \mu(y-\eta)\, e^{-\mu|z|}\, d\mu\right]\,. \tag{I. 9}$$

This form of integral representation for φ is the one given by von Karman

(Ref. 22). A translation of the origin in the z-direction then gives Eq. (10) of the text. Another convenient form of φ may be obtained by introducing the following transformation of variables:

$$\lambda = k \cos\theta , \qquad \mu = k \sin\theta . \tag{I. 10}$$

If one uses the relations:

$$J_o (k \sqrt{x^2 + y^2}) = \frac{2}{\pi}\int_0^{\pi/2} \cos(kx \cos\theta) \cos(ky \sin\theta)\, d\theta , \tag{I. 11}$$

and

$$\cos ky = k\int_0^{\infty} J_o (k \sqrt{\xi^2 + y^2})\, d\xi , \tag{I, 12}$$

then Eq. (I. 9) is converted to:

$$\varphi = \frac{\operatorname{sign} z}{4\pi}\int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_{-\infty}^{x} d\xi \int_0^{\infty} e^{-k|z|} J_o(k \sqrt{\xi^2 + (y-\eta)^2})\, k\, dk \; ; \tag{I. 13}$$

or,

$$\varphi = \frac{\operatorname{sign} z}{4\pi}\int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\infty} d\xi \int_0^{\infty} e^{-k|z|} J_o(k \sqrt{(x-\xi)^2 + (y-\eta)^2})\, k\, dk \tag{I. 14}$$

The choice of (I. 9) or (I. 14) is only a matter of convenience. A close form of φ may be deduced from (I. 14) by noting that

$$\int_0^{\infty} k\, e^{-k|z|} J_o(k\varpi)\, dk = \frac{|z|}{(\varpi^2 + z^2)^{3/2}} . \tag{I. 15}$$

Hence,

$$\varphi = \frac{z}{4\pi}\int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_0^{\infty} \left[(x-\xi)^2 + (y-\eta)^2 + z^2\right]^{-3/2} d\xi$$

$$= \frac{z}{4\pi}\int_{-\infty}^{\infty} \Gamma(\eta) \frac{1}{(y-\eta)^2 + z^2} \left(1 + \frac{x}{\sqrt{x^2 + (y-\eta)^2 + z^2}}\right) d\eta \tag{I. 16}$$

It may be remarked that these results may also be obtained from the differential point of view by considering the velocity field induced by an element of circulation $\Gamma(y)dy$ at y and by integrating it along the lifting line.

If $|\text{grad}\ \varphi| << U$ and the effect of gravity may be neglected, then the linearized pressure is given by

$$p - p_o = -\rho U \varphi_x \tag{I.17}$$

where from (I.16)

$$\varphi_x = \frac{z}{4\pi}\int_{-\infty}^{\infty} \Gamma(\eta) \frac{d\eta}{\left[x^2 + (y-\eta)^2 + z\right]^{3/2}} . \tag{I.18}$$

APPENDIX II

Formulation of the Nonstationary Flow Problem and the Stationary Flow as a Limiting Case

Take axes 0x and 0y in the undisturbed surface of deep water, and 0z vertically upward, the origin being at rest with respect to the fluid at infinity. Consider now the flow caused by a wing which starts to move at $t = 0$ from the position (0, 0, -h) in the negative x-direction with velocity U. The effect of an element of wing span $\Delta\eta$ at $y = \eta$ of circulation $\Gamma(\eta)\Delta\eta$ is to produce an initial impulse symmetrical about $(0, \eta, 0)$ on the initially still surface. These conditions correspond to initial data for $\varphi(x, y, z, t)$ and $\zeta(x, y, t)$ as follows:

$$\begin{cases} \rho\,\Delta\phi(x, y, 0, 0) = \Delta F(\varpi), \qquad \varpi^2 = x^2 + (y-\eta)^2 \\ \Delta\zeta(x, y, 0) = 0, \end{cases} \tag{II.1}$$

where $\Delta F(\varpi)$ should be equal to the initial pressure perturbation on the free surface. Using (I.17) and (I.18) and considering the effects due to both the wing and its image, we obtain

$$\Delta F(\varpi) = -\frac{\rho U h}{2\pi}\frac{\Gamma(\eta)\Delta\eta}{\left[x^2 + (y-\eta)^2 + h^2\right]^{3/2}} = -\frac{\rho U h}{2\pi}\frac{\Gamma(\eta)\Delta\eta}{\left[\varpi^2 + h^2\right]^{3/2}}, \tag{II.2}$$

The solution for φ which satisfies the Laplace equation with initial conditions (II.1) is (Ref. 5, p. 432):

$$\rho\,\Delta\phi = \int_0^\infty e^{kz} J_0(k\varpi)\cos(\sqrt{gk}\,t)\,k\,dk\int_0^\infty \Delta F(a)\,J_0(ka)\,a\,da, \qquad (z \leq 0)$$

and

$$\Delta\zeta = \frac{1}{g\rho}\int_0^\infty \sqrt{gk}\,\sin(\sqrt{gk}\,t)\,J_0(k\varpi)\,k\,dk\int_0^\infty \Delta F(a)\,J_0(ka)\,a\,da.$$

However,

$$\int_0^\infty \Delta F(a)\, J_0(ka)\, a\, da = -\frac{\rho U h}{2\pi}\,\Gamma'(\eta)\,\Delta\eta \int_0^\infty \frac{a\, J_0(ka)}{(a^2+h^2)^{3/2}}\, da$$

$$= -\frac{\rho U h}{2\pi}\,\Gamma'(\eta)\,\Delta\eta \sqrt{\frac{2k}{h\pi}}\; K_{-\frac{1}{2}}(kh) \qquad \text{(Ref. 26, p. 30)}$$

$$= -\frac{\rho U}{2\pi}\, e^{-kh}\,\Gamma'(\eta)\,\Delta\eta$$

so that

$$\Delta\phi = -\frac{U}{2\pi}\,\Gamma'(\eta)\,\Delta\eta \int_0^\infty e^{k(z-h)} \cos(\sqrt{gk}\, t)\, J_0(k\varpi)\, k\, dk$$

$$\Delta\zeta = -\frac{U}{2\pi g}\,\Gamma'(\eta)\,\Delta\eta \int_0^\infty e^{-hk} \sqrt{gk}\, \sin(\sqrt{gk}\, t)\, J_0(k\varpi)\, k\, dk$$

The resulting values of ϕ and ζ due to the entire wing may then be obtained by integrating $\Delta\phi$ and $\Delta\zeta$ along the span:

$$\left.\begin{aligned} \phi &= -\frac{U}{2\pi}\int_{-\infty}^{\infty} \Gamma'(\eta)\, d\eta \int_0^\infty e^{k(z-h)} \cos(\sqrt{gk}\, t)\, J_0(k\varpi)\, k\, dk \\ \zeta &= -\frac{U}{2\pi g}\int_{-\infty}^{\infty} \Gamma'(\eta)\, d\eta \int_0^\infty e^{-hk} \sqrt{gk}\, \sin(\sqrt{gk}\, t)\, J_0(k\varpi)\, k\, dk \end{aligned}\right\} \qquad \text{(II. 3)}$$

It should be pointed out here that these are only the results due to an impulsive motion of a hydrofoil. To extend them to the case of a continuously moving wing in $(-x)$-direction with velocity U, we have to superimpose all instantaneous disturbances at $x = -U\tau$ at time $t = \tau$. We replace t by $t-\tau$ and x by $x-\xi = x+U\tau$ and integrate with respect to τ over the time during which the system has been in motion.

$$\phi = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma'(\eta)\, d\eta \int_0^t U(\tau)\, d\tau \int_0^\infty e^{k(z-h)} \cos\sqrt{gk}\,(t-\tau)\, J_0\!\left(k\sqrt{(x+U\tau)^2+(y-\eta)^2}\right) k\, dk \qquad \text{(II. 4)}$$

$$\zeta = -\frac{1}{2\pi g}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_{0}^{t}U(\tau)\,d\tau\int_{0}^{\infty}e^{-kh}\sqrt{gk}\,\sin\sqrt{gk}(t-\tau)\,J_o\left(k\sqrt{(x+U\tau)^2+(y-\eta)^2}\right)k\,dk \tag{II.5}$$

These are the integral representations of the solution of nonstationary motion of a hydrofoil. This ϕ is only the part of the perturbation potential due to effects of free surface and wave formations, not including the contribution of the wing and its image.

Now we shall pass on to the limiting case such that the wing has been in motion since $\tau = -\infty$ with constant velocity U. A difficulty arises, however, because the integral with respect to τ from $-\infty$ to t is nonconvergent. Since the experimental evidence ensures that ζ should be finite, we must seek a remedy in order to agree with observation. Mathematically it is usual in these problems to introduce a convergence factor $e^{-\mu(t-\tau)}$ and then let μ tend to zero afterwards.

$$\phi = -\frac{U}{2\pi}\lim_{\mu\to 0}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_{-\infty}^{t}e^{-\mu(t-\tau)}d\tau\int_{0}^{\infty}e^{k(z-h)}\cos\sqrt{gk}(t-\tau)\,J_o\left(k\sqrt{(x+U\tau)^2+(y-\eta)^2}\right)k\,dk \tag{II.6}$$

$$\zeta = -\frac{U}{2\pi g}\lim_{\mu\to 0}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_{-\infty}^{t}e^{-\mu(t-\tau)}d\tau\int_{0}^{\infty}e^{-hk}\sqrt{gk}\,\sin\sqrt{gk}(t-\tau)\,J_o\left(k\sqrt{(x+U\tau)^2+(y-\eta)^2}\right)k\,d \tag{II.7}$$

Physically it is usually explained that the indeterminateness of the problem is due to the absence of viscous effect. To avoid this difficulty, Rayleigh suggested the assumption that the fluid element is also subject to a frictional resisting force proportional to the relative velocity. Then, instead of Eq. (4) we have (Ref. 5, p. 399)

$$\frac{p}{\rho} + \frac{\partial\varphi}{\partial t} + gz - \mu\phi = 0 \tag{II.8}$$

From this it can be shown that the circulation, hence also ϕ, has a factor $e^{-\mu t}$, which shows the damping due to viscous effect.

It is natural to expect that expressions (II. 6) and (II. 7) will become independent of t with respect to a coordinate system moving with the hydrofoil.

Hence we introduce the following Galilean transformation:

$$\bar{x} = x + Ut\,, \qquad \bar{y} = y\,, \qquad \bar{z} = z\,, \qquad \bar{t} = t \tag{II.9}$$

and change the variable of integration from τ to $T = t - \tau$ so that

$$x + U\tau = \bar{x} - UT$$

and then dropping the bar, we obtain:

$$\phi = -\frac{U}{2\pi}\lim_{\mu\to 0}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_0^{\infty}e^{-\mu T}dT\int_0^{\infty}e^{k(z-h)}\cos(\sqrt{gk}\,T)J_0\left(k\sqrt{(x-UT)^2+(y-\eta)^2}\right)k\,dk \tag{II.10}$$

$$\zeta = -\frac{U}{2\pi g}\lim_{\mu\to 0}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_0^{\infty}e^{-\mu T}dT\int_0^{\infty}e^{-hk}\sqrt{gk}\,\sin(\sqrt{gk}\,T)J_0\left(k\sqrt{(x-UT)^2+(y-\eta)^2}\right)k\,dk \tag{II.11}$$

which clearly show the independence of t. By interchanging the order of integration, and using the relation:

$$J_0\left(k\sqrt{x^2+y^2}\right) = \frac{1}{\pi}\int_0^{\pi}e^{ikx\cos\theta}\cos(ky\sin\theta)\,d\theta \tag{II.12}$$

we have

$$\begin{aligned}
\mathcal{J}_4 &\equiv \lim_{\mu\to 0}\int_0^{\infty}e^{-\mu T}\sin(\sqrt{gk}\,T)\,J_0\left(k\sqrt{(x-UT)^2+(y-\eta)^2}\right)k\,dk \\
&= \lim_{\mu\to 0}\frac{1}{\pi}\int_0^{\pi}e^{ikx\cos\theta}\cos(k(y-\eta)\sin\theta)\int_0^{\infty}e^{-(\mu+ikU\cos\theta)T}\sin(\sqrt{gk}\,T)\,dT \\
&= -\lim_{\mu\to 0}\frac{1}{\pi U^2}\sqrt{\frac{g}{k}}\int_0^{\pi}e^{ikx\cos\theta}\cos(k(y-\eta)\sin\theta)\frac{\sec^2\theta}{k-(\kappa\sec^2\theta+2i\frac{\mu}{U}\sec\theta)}\,d\theta
\end{aligned} \tag{II.13}$$

where $\kappa = g/U^2$ and in the last step terms of $O(\mu^2)$ have been neglected. Eq. (II.11) then becomes

$$\zeta = \frac{1}{2\pi^2 U}\lim_{\mu\to 0}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_0^{\pi}d\theta\int_0^{\infty}e^{-hk+ikx\cos\theta}\cos(k(y-\eta)\sin\theta)\frac{(\sec^2\theta)\,k\,dk}{k-(\kappa\sec^2\theta+2i\frac{\mu}{U}\sec\theta)} \tag{II.14}$$

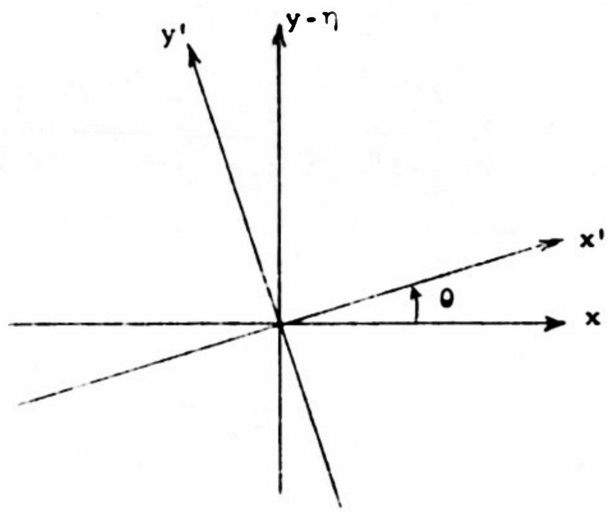
If we let $\mu \to 0$ before we evaluate these integrals, we then obtain the same expression for ζ as given by Eq. (16) which is derived by neglecting the viscosity from the beginning. On the other hand, it is interesting to see what the value of ζ will be as $x \to \pm\infty$ if we let $\mu \to 0$ after the evaluation of the integral with respect to k. First we rewrite (II. 14) as the real part of the following integral:

$$\zeta = \frac{R\ell}{2\pi^2 U} \lim_{\mu \to 0} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_{-\pi/2}^{\pi/2} \sec^2\theta\, d\theta \int_0^{\infty} e^{-hk} \left[\frac{e^{i(x\cos\theta + (y-\eta)\sin\theta)k}}{k - (\kappa \sec^2\theta + 2i\frac{\mu}{U}\sec\theta)} \right] k\, dk \tag{II. 15}$$

This expression of ζ indicates very clearly that the surface elevation consists of plane wave elements, each component moving in a line making an angle θ with $0x$. Now we rotate the axes $0x$, $0y$ to $0x'$, $0y'$ given by

$$x' = x\cos\theta + (y-\eta)\sin\theta, \qquad y' = -x\sin\theta + (y-\eta)\cos\theta \tag{II. 16}$$

as shown in the following figure.



In order to study the behavior of ζ as $x' \to \pm\infty$, the integral with respect to k may be transformed by contour integration considering k to be complex. For $x' < 0$, we can deform the contour to the negative imaginary axis of k; and for $x' > 0$, to the positive imaginary axis. Note that the integrand has a simple pole at $k = \kappa\sec^2\theta + i\frac{2\mu}{U}\sec\theta$ which is located inside the contour for $x' > 0$ when $\mu \neq 0$. The residue of the integrand at this pole is equal to $e^{-h\kappa\sec^2\theta + ix'\kappa\sec^2\theta}\, \kappa\sec^2\theta$ as $\mu \to 0$ and should only concern the integral $x' > 0$. By making μ zero after the deformation has been carried out, we

then obtain different representations for $x' < 0$ and $x' > 0$ corresponding to different behaviors upstream and downstream of each wave element:

$$\lim_{\mu \to 0} R\ell \int_0^{\infty} e^{-hk + ix'k} \frac{k}{k - (\kappa \sec^2 \theta + 2i \frac{\mu}{U} \sec\theta)} dk$$

$$= -2\pi\kappa \sec^2\theta\, e^{-h\kappa\sec^2\theta} \sin(\kappa x' \sec^2\theta) + \int_0^{\infty} e^{-x'a} \frac{\kappa \sec^2\theta \cos ah + a \sin ah}{a^2 + \kappa^2 \sec^4\theta}\, a\, da, \quad x' > 0;$$

$$= \int_0^{\infty} e^{x'a} \frac{\kappa \sec^2\theta \cos ah + a \sin ah}{a^2 + \kappa^2 \sec^4\theta}\, a\, da, \quad x' < 0. \qquad \text{(II. 17)}$$

The first term in (II. 17) represents simple waves on the downstream side of the wave front $x' = x\cos\theta + (y-\eta)\sin\theta = 0$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$ while the other terms represent disturbances symmetrical with respect to $x' = 0$ and diminishing exponentially with increasing distance from $x' = 0$. It should also be pointed out here that the integration with respect to η (or θ) should be divided into two regions in which the values of η correspond to $x' > 0$ and $x' < 0$ respectively for an assigned point (x, y). The final expression of ζ is then

$$\zeta = \frac{1}{2\pi^2 U} \int_0^{\pi/2} \sec^2\theta\, d\theta \left\{ \int_{-\infty}^{y + x\cot\theta} \Gamma(\eta)\, d\eta \left[-2\pi\kappa e^{-h\kappa\sec^2\theta} \sec^2\theta \sin(\kappa x' \sec^2\theta) \right.\right.$$

$$\left. + \int_0^{\infty} e^{-x'a} \frac{\kappa \sec^2\theta \cos ah + a \sin ah}{a^2 + \kappa^2 \sec^4\theta}\, a\, da \right]$$

$$+ \int_{y + x\cot\theta}^{0} \Gamma(\eta)\, d\eta \int_0^{\infty} e^{x'a} \frac{\kappa \sec^2\theta \cos ah + a \sin ah}{a^2 + \kappa^2 \sec^4\theta}\, a\, da$$

$$+ \int_{y - x\cot\theta}^{\infty} \Gamma(\eta)\, d\eta \left[-2\pi\kappa e^{-h\kappa\sec^2\theta} \sec^2\theta \sin(\kappa \bar{x}' \sec^2\theta) \right.$$

$$\left. + \int_0^{\infty} e^{-\bar{x}'a} \frac{\kappa \sec^2\theta \cos ah + a \sin ah}{a^2 + \kappa^2 \sec^4\theta}\, a\, da \right]$$

$$\left. + \int_{-\infty}^{y - x\cot\theta} \Gamma(\eta)\, d\eta \int_0^{\infty} e^{\bar{x}'a} \frac{\kappa \sec^2\theta \cos ah + a \sin ah}{a^2 + \kappa^2 \sec^4\theta}\, a\, da \right\} \qquad \text{(II. 18)}$$

where

$$\bar{x}' = x\cos\theta - (y-\eta)\sin\theta\ . \qquad \text{(II. 19)}$$

In this way we may study the behavior of ζ as $x \to +\infty$. The second and fourth integrals with respect to η are negligibly small for large positive x and finite fixed y as can be seen from the fact that both upper and lower limits of integration tend to $+\infty$ and $-\infty$ respectively. The same is true for terms with $e^{-x'a}$ and $e^{-\bar{x}'a}$ which may also be neglected. Hence the only significant term in Eq. (II. 18) for large positive x is

$$\lim_{x\to+\infty} \zeta \cong -\frac{2K}{\pi U}\int_{-\infty}^{\infty}\Gamma(\eta)\int_{0}^{\pi/2} e^{-2hK\sec^2\theta}\sec^4\theta\, d\theta\ \sin(K x\cos\theta)\cos(K(y-\eta)\sec^2\theta\sin\theta)\, d\eta$$

$$\text{(II. 20)}$$

which is the same as Eq. (20). The limiting value of the second integral goes to zero as $0(\frac{1}{\sqrt{x}})$ which can be shown by applying the method of stationary phase.

$$\zeta = -\frac{1}{U^2}\sqrt{\frac{2g}{\pi x}}\ \sin(Kx + \frac{\pi}{4})\int_{-\infty}^{\infty}\Gamma(\eta)\, d\eta + 0(\frac{1}{x})\ , \quad \text{as } x\to+\infty\ , \text{ and } y \text{ fixed, finite}$$

$$\text{(II. 21)}$$

For x negative and large with y again finite and fixed, the first and third integrals are negligibly small. The sum of the other two integrals can be shown by using Watson's lemma to be much less than the absolute value of ζ given by Eq. (II. 21).

It is of interest to note that by considering a convergence factor μ (usually called Rayleigh's viscous term) in the intermediate stages, the final result of surface elevation becomes asymmetrical with respect to x: ζ damps out like $0(\frac{1}{x^3})$ on the upstream side and has a wave formation downstream of the hydrofoil and finally diminishes like $0(\frac{1}{\sqrt{x}})$ for x large.

APPENDIX III

Calculation of Wave Resistance by Method of Traveling Pressure

According to the method of traveling surface pressure (cf. Roy. Soc. Proc. A, Vol. 93, p. 244 (1917)), the wave resistance is simply the total resolved surface pressure in the x-direction. Because ζ is everywhere small, this leads to

$$R = \iint F(\varpi) \frac{\partial \zeta}{\partial x} dx\, dy \tag{III. 1}$$

taken over the whole surface. $F(\varpi)$ is given by (II. 2) to be

$$F(\varpi) = -2\rho U (\varphi_{1x})_{z=0} = -\frac{\rho U h}{2\pi}\int_{-\infty}^{\infty} \Gamma(\eta) \frac{d\eta}{\left[x^2+(y-\eta)^2+h^2\right]^{3/2}} \tag{III. 2}$$

Since $\left[x^2+(y-h)^2+h^2\right]^{3/2}$ is even in x and $(y-\eta)$, it follows that only the part of $\frac{\partial \zeta}{\partial x}$ even in both x and $(y-\eta)$ contributes to R. Using the form of ζ given by Eq. (18), which is the only part even in x and $(y-\eta)$, we have

$$\frac{\partial \zeta}{\partial x} = -\frac{\kappa^2}{U\pi}\int_{-\infty}^{\infty}\Gamma(\eta')\,d\eta'\int_0^{\pi/2} \sec^5\theta\, e^{-h\kappa\sec^2\theta}\cos(\kappa x\sec\theta)\cos(\kappa(y-\eta')\sec^2\theta\sin\theta)\,d\theta$$

Hence

$$R = \frac{\rho h \kappa^2}{\pi^2}\int_{-\infty}^{\infty}\Gamma(\eta)\,d\eta\int_{-\infty}^{\infty}\Gamma(\eta')\,d\eta'\int_0^{\pi/2}\sec^5\theta\, e^{-h\kappa\sec^2\theta}\,d\theta$$

$$\int_{-\infty}^{\infty}dy\int_0^{\infty}dx\,\frac{\cos(\kappa x\sec\theta)\cos(\kappa(y-\eta')\sec^2\theta\sin\theta)}{\left[x^2+(y-\eta)^2+h^2\right]^{3/2}}$$

Now the integrals with respect to x and y can be carried out by introducing the transformation:

$$x = r\cos\phi \qquad y-\eta = r\sin\phi$$

and using the notation

$$a = \kappa\sec\theta \qquad b = \kappa\sec^2\theta\sin\theta$$

We then have

$$\int_{-\infty}^{\infty} dy \int_{0}^{\infty} \frac{\cos(ax)\cos(b(y-\eta'))}{(x^2+(y-\eta)^2+h^2)^{3/2}}\, dx$$

$$= \int_{0}^{\infty} \frac{r\,dr}{(r^2+h^2)^{3/2}} \int_{-\pi/2}^{\pi/2} \cos(ar\cos\phi)\left[\cos(br\sin\phi)\cos(b(\eta'-\eta)) + \sin(br\sin\phi)\sin(b(\eta'-\eta))\right] d\phi$$

$$= 2\cos(b(\eta'-\eta)) \int_{0}^{\infty} \frac{r}{(r^2+h^2)^{3/2}}\, dr \int_{0}^{\pi/2} \cos(ar\cos\phi)\cos(br\sin\phi)\, d\phi$$

$$= \pi\cos(b(\eta'-\eta)) \int_{0}^{\infty} \frac{r}{(r^2+h^2)^{3/2}}\, J_0(\kappa r\sec^2\theta)\, dr \qquad \text{(cf. Eq. (II. 12))}$$

$$= \left(\frac{2\pi\kappa\sec^2\theta}{h}\right)^{\frac{1}{2}} \cos(b(\eta'-\eta))\, K_{1/2}(\kappa h\sec^2\theta) \qquad \text{(cf. Ref. 26, p. 30)}$$

$$= \frac{\pi}{h}\, e^{-h\kappa\sec^2\theta} \cos(\kappa(\eta'-\eta)\sec^2\theta\sin\theta)$$

Therefore we obtain

$$R = \frac{\rho\kappa^2}{\pi} \int_{-\infty}^{\infty} \Gamma(\eta)\, d\eta \int_{-\infty}^{\infty} \Gamma(\eta')\, d\eta' \int_{0}^{\pi/2} \sec^5\theta\, e^{-2h\kappa\sec^2\theta} \cos(\kappa(\eta-\eta')\sec^2\theta\sin\theta)\, d\theta$$

$$= \pi\rho\kappa^2 \int_{+0}^{\pi/2} e^{-2h\kappa\sec^2\theta} \sec^5\theta \left\{\left[f(\kappa\sec^2\theta\sin\theta)\right]^2 + \left[g(\kappa\sec^2\theta\sin\theta)\right]^2\right\} d\theta$$

where f and g are defined by Eq. (27). This result agrees with Eq. (29).

APPENDIX IV

Evaluation of Some Integrals

(A) Derivation of the Integral in Eq. (46)

Substituting Eq. (45) in (43), integrating by parts with respect to η and using the relations

$$y = b\cos\phi \qquad \eta = b\cos\psi \quad , \tag{IV. 1}$$

we have

$$u(y) = -\frac{1}{2\pi^2}\int_{-b}^{b}\Gamma(\eta)\cos(\mu\eta\sin\theta)\,d\eta\int_0^{\pi/2}d\theta\int_0^{\infty}e^{-2h\mu}\cos(\mu y\sin\theta)\,\frac{\mu+\kappa\sec^2\theta}{\mu-\kappa\sec^2\theta}\,\mu\,d\mu$$

$$= \frac{1}{2\pi^2}\int_{-b}^{b}\sin(\mu\eta\sin\theta)\frac{d\Gamma(\eta)}{d\eta}\,d\eta\int_0^{\pi/2}\frac{d\theta}{\sin\theta}\int_0^{\infty}e^{-2h\mu}\cos(\mu y\sin\theta)\,\frac{\mu+\kappa\sec^2\theta}{\mu-\kappa\sec^2\theta}\,d\mu$$

$$= -\frac{2bU}{\pi^2}\sum_{n=0}^{\infty}(2n+1)\,A_{2n+1}\int_0^{\pi}\sin(\mu b\sin\theta\cos\psi)\cos(2n+1)\psi\,d\psi\int_0^{\pi/2}\frac{d\theta}{\sin\theta}\,\cdot$$

$$\cdot\int_0^{\infty}e^{-2h\mu}\cos(\mu y\sin\theta)\,\frac{\mu+\kappa\sec^2\theta}{\mu-\kappa\sec^2\theta}\,d\mu$$

The interchanging of the order of $\sum$ and $\int$ sign is justified if the resulting series converges. Noting that

$$\int_0^{\pi}\sin(z\cos\psi)\cos(2n+1)\psi\,d\psi = (-)^n\pi\,J_{2n+1}(z) \tag{IV. 2}$$

where J_{2n+1} denotes Bessel function of the first kind (cf. Ref. 24, p. 20), we finally obtain

$$u(y) = -\frac{2U}{\pi}\sum_{n=0}^{\infty}(-)^n(2n+1)\,A_{2n+1}\int_0^{\pi/2}\frac{d\theta}{\sin\theta}\int_0^{\infty}e^{-2\lambda t}\,\frac{t+b\kappa\sec^2\theta}{t-b\kappa\sec^2\theta}\,J_{2n+1}(t\sin\theta)\,\cdot$$

$$\cdot\cos(t\cos\phi\sin\theta)\,dt \tag{IV. 3}$$

which is Eq. (46).

(B) Derivation of the Integral in Eq. (47)

Substituting Eq. (45) into (44) and proceeding in a way similar to that in the previous case, we have

$$w(y) = -\frac{1}{4\pi}\int_{-b}^{b} \sin\mu\eta \frac{d\Gamma}{d\eta}\, d\eta \int_0^{\infty} (1 - e^{-2h\mu}) \cos\mu y \, d\mu$$

$$- \frac{\kappa}{\pi}\int_{-b}^{b} \sin(\kappa\eta\sec^2\theta \sin\theta)\frac{d\Gamma}{d\eta}\, d\eta \int_0^{\pi/2} e^{-2h\kappa\sec^2\theta} \cos(\kappa y \sec^2\theta \sin\theta) \frac{\sec^3\theta}{\sin\theta}\, d\theta$$

$$= U \sum_{n=0}^{\infty} (-)^n (2n+1) A_{2n+1} \Bigg\{ \int_0^{\infty} (1 - e^{-2\lambda t}) J_{2n+1}(t) \cos(t\cos\phi)\, dt$$

$$+ 4b\kappa \int_0^{\pi/2} e^{-2h\kappa\sec^2\theta} J_{2n+1}(b\kappa\sec^2\theta \sin\theta) \cos(b\kappa\cos\phi \sec^2\theta \sin\theta) \frac{\sec^3\theta}{\sin\theta}\, d\theta \Bigg\} .$$

(IV. 4)

The first integral on the right hand side can be integrated to give (cf. Ref. 25, p. 37 and p. 33):

$$\int_0^{\infty} J_{2n+1}(t) \cos(t\cos\phi)\, dt = \frac{\cos\left[(2n+1)\sin^{-1}\cos\phi\right]}{\sin\phi} = (-)^n \frac{\sin(2n+1)\phi}{\sin\phi}$$

(IV. 5)

and

$$\int_0^{\infty} e^{-2\lambda t} J_{2n+1}(t) \cos(t\cos\phi)\, dt = R\ell \int_0^{\infty} e^{-(2\lambda - i\cos\phi)t} J_{2n+1}(t)\, dt$$

$$= R\ell \left\{ \left[\sqrt{1+(2\lambda - i\cos\phi)^2} - (2\lambda - i\cos\phi)\right]^{2n+1} \left[1+(2\lambda - i\cos\phi)^2\right]^{-\frac{1}{2}} \right\}$$

(IV. 6)

Combining Eqs. (IV.4-6), we obtain Eq. (47).

(C) Evaluation of the Integral in Eq. (61)

$$\int_0^{\infty} e^{-2h\mu}\, \frac{J_1^2(\mu b)}{\mu}\, d\mu = \int_0^{\infty} e^{-2\lambda t}\, J_1^2(t)\, t^{-1}\, dt \qquad (\lambda = \frac{h}{b})$$

$$= -\frac{1}{2}\frac{\partial}{\partial\lambda}\int_0^{\infty} e^{-2\lambda t}\, J_1^2(t)\, t^{-2}\, dt$$

$$= \frac{2}{\pi}\frac{\partial}{\partial\lambda}\int_0^{\pi/2}\left[\lambda - \sqrt{\lambda^2+\sin^2\theta}\,\right]\cos^2\theta\, d\theta \qquad \text{(Ref. 24, p. 389)}$$

$$= \frac{1}{2}\left\{1 - \frac{4}{\pi}\lambda\int_0^{\pi/2}\frac{\cos^2\theta}{\sqrt{\lambda^2+\sin^2\theta}}\, d\theta\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{4}{\pi}\lambda\sqrt{1+\lambda^2}\left[K\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - E\left(\frac{1}{\sqrt{1+\lambda^2}}\right)\right]\right\} \qquad \text{(Ref. 26, p. 73)} \qquad \text{(IV. 7)}$$

where K and E denote the complete elliptic integral of the first and second kind respectively.

(D) Derivation of the Result of Eq. (64)

Applying the transformation $\sec\theta = \cosh\frac{u}{2}$ to the integral in Eq. (63), we have

$$\mathcal{I}_1 \equiv \int_0^{\pi/2} e^{-(2/\sigma)\sec^2\theta}\left[\frac{J_1(\frac{1}{\beta}\sec^2\theta\,\sin\theta)}{\sec^2\theta\,\sin\theta}\right]^2 \sec^5\theta\, d\theta$$

$$= \frac{1}{2}e^{-1/\sigma}\int_0^{\infty} e^{-(1/\sigma)\cosh u}\left[\frac{J_1(\frac{1}{2\beta}\sinh u)}{\frac{1}{2}\sinh u}\right]^2\left(\frac{1+\cosh u}{2}\right)^2 du\,. \qquad \text{(IV. 8)}$$

Note that the above integral still converges as $\frac{1}{\sigma}\to 0$, keeping β constant. One way to obtain the result given in Eq. (64) is by expanding the term J_1^2 into an infinite series (cf. Ref. 24, p. 147) as follows:

$$J_1^2(z) = \sum_{n=0}^{\infty}\frac{(-)^n\,\Gamma(2n+3)\,(\frac{z}{2})^{2n+2}}{n!\,(n+2)!\,[(n+1)!]^2} = \sum_{n=0}^{\infty}\frac{(-)^n\,\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\,n!\,(n+1)!\,(n+2)!}\, z^{2n+2}\,.$$

Introducing this expansion to Eq. (IV.8) and interchanging the order of $\int$ and $\sum$ sign, we obtain

$$\mathcal{J}_1 = \frac{1}{8\sqrt{\pi}} \frac{e^{-\frac{1}{\sigma}}}{\beta^2} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma(n+\frac{3}{2})}{n!(n+1)!(n+2)!} \left(\frac{1}{2\beta}\right)^{2n} \int_0^{\infty} e^{-\frac{1}{\sigma}\cosh u} \sinh^{2n} u \,(2 + \sinh^2 u + 2\cosh u)\, du$$

provided that the resulting series converges absolutely. The region of convergence for σ and λ will be shown below. By using the following relations (cf. Ref. 24, p. 172)

$$\int_0^{\infty} e^{-a\cosh u} \sinh^{2n} u\, du = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}} \left(\frac{2}{a}\right)^n K_n(a)\,, \qquad (n > -\tfrac{1}{2})\,,\ a > 0)$$

$$\int_0^{\infty} e^{-a\cosh u} \sinh^{2n} u \cosh u\, du = -\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}\, 2^n \frac{d}{da}\left(\frac{K_n(a)}{a^n}\right) = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}} \left(\frac{2}{a}\right)^n K_{n+1}(a) \tag{IV.9}$$

where K_n denotes modified Bessel function of the second kind, we obtain

$$\mathcal{J}_1 = \frac{e^{-\frac{1}{\sigma}}}{4\pi(\beta)^2} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!(n+1)!(n+2)!} \left(\frac{\sigma}{2\beta^2}\right)^n \left\{ K_n\left(\frac{1}{\sigma}\right) + \left[1+\sigma(n+\tfrac{1}{2})\right] K_{n+1}\left(\frac{1}{\sigma}\right) \right\}. \tag{IV.10}$$

It can be shown that the above series has different regions of convergence for large and small values of σ due to the different behavior of the function $K_n(\frac{1}{\sigma})$. When σ is small, this series converges absolutely and uniformly for any close interval of β; but for large values of σ ($\gg 1$), the above series only converges for $\frac{\beta}{\sigma} = \lambda > 1$. The interchanging of the $\int$ and $\sum$ operation is justified only when the above conditions are fulfilled.

(E) Derivation of the Result of Eq. (68)

We want to sum the series in Eq. (IV.10) for large values of σ. In this case $K_n(\frac{1}{\sigma})$ has the following expansion:

$$
\left.\begin{aligned}
K_0\left(\frac{1}{\sigma}\right) &= \log(2\sigma)\; I_0\left(\frac{1}{\sigma}\right) + \sum_{m=0}^{\infty} \frac{(2\sigma)^{-2m}}{(m!)^2}\, \psi(m+1)\,, \\
K_n\left(\frac{1}{\sigma}\right) &= \frac{1}{2} \sum_{m=0}^{n-1} \frac{(-)^m (n-m-1)!}{m!} (2\sigma)^{n-2m} + (-)^n \sum_{m=0}^{\infty} \frac{(2\sigma)^{-n-2m}}{m!\,(n+m)!} \Big[\log(2\sigma) \\
&\qquad + \frac{1}{2}\,\psi(m+1) + \frac{1}{2}\,\psi(n+m+1) \Big]\,, \quad (n \geq 1)
\end{aligned}\right\} \quad \text{(IV. 11)}
$$

It is easy to see that the most important contribution to the sum of the series in Eq. (IV. 10) comes from the term $\frac{1}{2}(n!)(2\sigma)^{n+1}$ of the expansion of $K_{n+1}(\frac{1}{\sigma})$. The detail of the calculation can be shown as follows:

Decompose $\mathcal{J}_1$ into three parts such that

$$
\mathcal{J}_1 = \mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{13}\,, \qquad \text{(IV. 12)}
$$

where $\mathcal{J}_{11}$, $\mathcal{J}_{12}$ and $\mathcal{J}_{13}$ are given below.

$$
\begin{aligned}
\mathcal{J}_{11} &= \frac{e^{-\frac{1}{\sigma}}}{4\pi\beta^2} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2})}{n!\,(n+1)!\,(n+2)!} \left(\frac{\sigma}{2\beta^2}\right)^n \left[\sigma(n+\tfrac{1}{2})\right] K_{n+1}\left(\frac{1}{\sigma}\right) \\
&\doteq \frac{e^{-\frac{1}{\sigma}}}{4\pi} \sum_{n=0,1}^{\infty} \frac{(-)^n \Gamma(n+\frac{3}{2}) \Gamma(n+\frac{3}{2})}{n!\,(n+1)!\,(n+2)!} \left(\frac{\sigma}{2\beta^2}\right)^{n+1} \\
&\qquad \left\{ (n!)(2\sigma)^{n+1} - (n-1)!\,(2\sigma)^{n-1} + 2(-)^{n+1} \frac{\log(2\sigma)}{n!\,(2\sigma)^{n+1}} \right\} \\
&\doteq \frac{e^{-\frac{1}{\sigma}}}{4\pi} \left\{ \sum_{n=0}^{\infty} \frac{(-)^n \Gamma^2(n+\frac{3}{2})}{(n+1)!\,(n+2)!} \left(\frac{1}{\lambda^2}\right)^{n+1} - \frac{1}{4\beta^2} \sum_{n=1}^{\infty} \frac{(-)^n \Gamma^2(n+\frac{3}{2})}{n\,(n+1)!\,(n+2)!} \left(\frac{1}{\lambda^2}\right)^n \right. \\
&\qquad \left. - \frac{\pi}{16\beta^2} \log(2\sigma) \right\}
\end{aligned} \qquad \text{(IV. 13)}
$$

Now both these two series converge only when $\lambda \geq 1$; but the first series can be directly related to a hypergeometric function which may then be continued

analytically to the region $0 \leq \lambda \leq 1$ as follows:

$$\sum_{n=0}^{\infty} \frac{(-)^n \Gamma^2(n+\frac{3}{2})}{(n+1)!\,(n+2)!} \left(\frac{1}{\lambda^2}\right)^{n+1} = \frac{\Gamma^2(\frac{1}{2})}{\Gamma(2)} - \sum_{s=0}^{\infty} \frac{(-)^s \Gamma^2(s+\frac{1}{2})}{\Gamma(s+2)} \frac{\left(\frac{1}{\lambda^2}\right)^s}{s!}$$

$$= \pi \left[1 - F\left(\tfrac{1}{2}, \tfrac{1}{2}; 2; -\tfrac{1}{\lambda^2}\right) \right]$$

$$= \pi \left\{ 1 - \frac{\lambda}{\sqrt{1+\lambda^2}} F\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; \frac{1}{1+\lambda^2}\right) \right\} .$$

If we use one of Gauss's recursion formulas, the above hypergeometric function can be expressed in terms of complete elliptic integrals, (cf. Ref. 26, pp. 9, 10).

$$F\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; k^2\right) = \frac{2}{k^2} \left[F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right) - F\left(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right) \right]$$

$$= \frac{4}{\pi} k^{-2} \left[K(k) - E(k) \right] .$$

Finally we have

$$\sum_{n=0}^{\infty} \frac{(-)^n \Gamma^2(n+\frac{3}{2})}{(n+1)!\,(n+2)!} \left(\frac{1}{\lambda^2}\right)^{n+1} = \pi \left\{ 1 - \frac{4}{\pi} \lambda \sqrt{1+\lambda^2} \left[K\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - E\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \right] \right\} .$$

(IV. 14)

To carry out the summation of the second series in Eq. (IV. 13), we note that if we write

$$f(z) = \sum_{n=1}^{\infty} \frac{(-)^n \Gamma^2(n+\frac{3}{2})}{n(n+1)!\,(n+2)!} z^{-n} , \qquad z = \lambda^2 , \tag{IV. 15}$$

then $f'(z)$ may be expressed in terms of a hypergeometric function together with an algebraic function as follows:

$$f'(z) = - \sum_{n=1}^{\infty} \frac{(-)^n \Gamma^2(n+\frac{3}{2})}{(n+1)!\,(n+2)!} z^{-(n+1)} = - \frac{\Gamma^2(\frac{1}{2})}{\Gamma(2)} \left[1 - \frac{1}{8}\frac{1}{z} - F\left(\tfrac{1}{2}, \tfrac{1}{2}; 2; -\tfrac{1}{z}\right) \right]$$

$$= -\pi \left\{ 1 - \frac{1}{8z} - \frac{4}{\pi} \sqrt{z(1+z)} \left[K\left(\frac{1}{\sqrt{1+z}}\right) - E\left(\frac{1}{\sqrt{1+z}}\right) \right] \right\} . \tag{IV. 16}$$

As $z = \lambda^2 \to 0$, $\frac{1}{\sqrt{1+z}} \to 1$; the asymptotic value of $f'(z)$ as $z \to 0$ may be deduced from the above relation by using the known asymptotic expansions of K and E.

$$f'(z) \cong \pi \left\{ -1 + \frac{1}{8z} + \frac{4}{\pi} \sqrt{\frac{z}{1+z}} \left[\log 4 \sqrt{\frac{1+z}{z}} - 1 + \frac{3}{4} \frac{z}{\sqrt{1+z}} (\log 4 \sqrt{\frac{1+z}{z}} - \frac{4}{3}) + 0(z^2 \log z) \right] \right\}$$

Since the integral of an asymptotic expression is also asymptotic, the asymptotic value of $f(z)$ can then be obtained by integrating the above equation. The final result is

$$f(z) \cong + \frac{\pi}{8} \log z + 0(1) = + \frac{\pi}{4} \log \frac{\beta}{\sigma} + 0(1) \qquad \text{(IV. 17)}$$

Therefore, substituting Eqs. (IV. 14, 17) into (IV. 13), we obtain:

$$\mathcal{J}_{11} \cong \frac{1}{4} \left\{ 1 - \frac{4}{\pi} \lambda \sqrt{1+\lambda^2} \left[K(\frac{1}{\sqrt{1+\lambda^2}}) - E(\frac{1}{\sqrt{1+\lambda^2}}) \right] - \frac{1}{16\beta^2} \log \beta + 0(\frac{1}{\beta^2}, \frac{1}{\sigma}) \right\}. \qquad \text{(IV. 18)}$$

The second part of $\mathcal{J}_1$, namely, $\mathcal{J}_{12}$, is given by

$$\mathcal{J}_{12} \equiv \frac{e^{-\frac{1}{\sigma}}}{4\pi\beta^2} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2})}{n!(n+1)!(n+2)!} (\frac{\sigma}{2\beta^2})^n K_{n+1}(\frac{1}{\sigma})$$

$$\doteq \frac{e^{-\frac{1}{\sigma}}}{4\pi\sigma} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2}) \Gamma(n+\frac{3}{2})}{n!(n+1)!(n+2)!} (\frac{\sigma}{2\beta^2})^{n+1} \left[(n!)(2\sigma)^{n+1} \right]$$

$$\doteq \frac{e^{-\frac{1}{\sigma}}}{4\pi\sigma} \left[\frac{\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} - \sum_{s=0}^{\infty} \frac{(-)^s \Gamma(s-\frac{1}{2}) \Gamma(s+\frac{1}{2})}{\Gamma(s+2)} \frac{(\frac{1}{\lambda^2})^s}{s!} \right]$$

$$= - \frac{e^{-\frac{1}{\sigma}}}{2\sigma} \left[1 - F(-\frac{1}{2}, \frac{1}{2}; 2; -\frac{1}{\lambda^2}) \right]$$

$$= \frac{e^{-\frac{1}{\sigma}}}{2\beta} \left[\sqrt{1+\lambda^2} F(-\frac{1}{2}, \frac{3}{2}; 2; \frac{1}{1+\lambda^2}) - \lambda \right] \qquad \text{(IV. 19)}$$

After this analytic continuation, if one uses again Gauss's recursion formula, one finds,

$$F(-\tfrac{1}{2}, \tfrac{3}{2}; 2; \frac{1}{1+\lambda^2}) = \frac{2}{3}\lambda^2 \left\{ F(\tfrac{1}{2}, \tfrac{3}{2}; 1; \frac{1}{1+\lambda^2}) - F(-\tfrac{1}{2}, \tfrac{3}{2}; 1; \frac{1}{1+\lambda^2}) \right\}$$

$$= \frac{2}{3} \left\{ \frac{2}{\pi}(1+\lambda^2)\, E(\frac{1}{\sqrt{1+\lambda^2}}) - \lambda^2 F(-\tfrac{1}{2}, \tfrac{3}{2}; 1; \frac{1}{1+\lambda^2}) \right\} \quad \text{(IV. 20)}$$

Combining Eqs. (IV. 19) and (IV. 20), we have

$$\mathcal{J}_{12} = \frac{1}{3\beta} \left\{ \frac{2}{\pi}(1+\lambda^2)^{3/2} E(\frac{1}{\sqrt{1+\lambda^2}}) - \frac{3}{2}\lambda - \lambda^2 \sqrt{1+\lambda^2}\, F(-\tfrac{1}{2}, \tfrac{3}{2}; 1; \frac{1}{1+\lambda^2}) \right\} (1+0(\frac{1}{\sigma}))$$

(IV. 21)

It is evident that the hypergeometric function in the above equation has singularity at $\lambda = 0$. To calculate its value near $\lambda = 0$, we need the following formula

$$F(a, b; a+b; z) = \sum_{n=o}^{\infty} a_n (1-z)^n \left[\beta_n + \log(1-z) \right] \quad \text{(IV. 22)}$$

where

$$a_n = - \frac{\Gamma(a+b)\Gamma(a+n)\Gamma(b+n)}{\Gamma^2(a)\Gamma^2(b)\Gamma^2(1+n)} ,$$

$$\beta_n = \psi(a+n) + \psi(b+n) - 2\psi(1+n) .$$

Hence

$$F(-\tfrac{1}{2}, \tfrac{3}{2}; 1; \frac{1}{1+\lambda^2}) = \frac{2}{\pi}\left\{ (2 - \log \frac{4\sqrt{1+\lambda^2}}{\lambda}) - \frac{3}{4\pi}(\frac{1}{3} - \log \frac{4\sqrt{1+\lambda^2}}{\lambda})(\frac{\lambda^2}{1+\lambda^2}) \right\} + 0(\lambda^4 \log \lambda),$$

(IV. 23)

and finally we have

$$\mathcal{J}_{12} \cong \frac{2}{3\pi\beta} \left\{ (1+\lambda^2)^{3/2} - \frac{3\pi}{4}\lambda - \frac{3}{2}\lambda^2 \sqrt{1+\lambda^2} \log \frac{4\sqrt{1+\lambda^2}}{\lambda} + 0(\lambda^2) \right\}. \quad \text{(IV. 24)}$$

The third part of $\mathcal{J}_1$ denoted by $\mathcal{J}_{13}$, is

$$\mathcal{J}_{13} = \frac{e^{-\frac{1}{\sigma}}}{4\pi\beta^2} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!(n+1)!(n+2)!} \left(\frac{\sigma}{2\beta^2}\right)^n K_n\left(\frac{1}{\sigma}\right)$$

$$\doteq \frac{e^{-\frac{1}{\sigma}}}{4\pi\beta^2} \left\{ \sum_{n=1}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!(n+1)!(n+2)!} \left(\frac{\sigma}{2\beta^2}\right)^n \left[\frac{1}{2}(n-1)!\,(2\sigma)^n \right] + \frac{\pi}{4} \log 2\sigma \right\},$$

or,

$$\mathcal{J}_{13} \doteq \frac{e^{-\frac{1}{\sigma}}}{8\pi\beta^2} \left\{ \sum_{n=1}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n\,(n+1)!\,(n+2)!} \left(\frac{1}{\lambda^2}\right)^n + \frac{\pi}{2} \log 2\sigma \right\}.$$

The series in the above equation can be approximated in a way similar to that applied to Eq. (IV.15). The final result is

$$\mathcal{J}_{13} \doteq \frac{1}{16\beta^2} \log\beta + 0\left(\frac{1}{\beta^2}, \frac{1}{\sigma}\right). \tag{IV.25}$$

Therefore, substituting Eqs. (IV.18), (IV.21) and (IV.25) into Eq. (IV.12), we obtain a good approximation of the value of $\mathcal{J}_1$ defined by (IV.8) or (IV.10) for σ large as follows:

$$\mathcal{J}_1 = \int_0^{\pi/2} e^{-\frac{2}{\sigma}\sec^2\theta} \left[\frac{J_1(\frac{1}{\beta})\sec^2\theta\sin\theta}{\sec^2\theta\sin\theta}\right]^2 \sec^5\theta\, d\theta$$

$$\doteq \frac{1}{4}\left\{1 - \frac{4}{\pi}\lambda\sqrt{1+\lambda^2}\left[K\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - E\left(\frac{1}{\sqrt{1+\lambda^2}}\right)\right]\right\} + \frac{1}{3\beta}\left\{\frac{2}{\pi}(1+\lambda^2)^{3/2} E\left(\frac{1}{\sqrt{1+\lambda^2}}\right) - \right.$$

$$\left. - \frac{3}{2}\lambda - \lambda^2\sqrt{1+\lambda^2}\, F\left(-\frac{1}{2}, \frac{3}{2}; 1; \frac{1}{1+\lambda^2}\right)\right\} + \frac{3}{64\beta^2}\log\beta + 0\left(\frac{1}{\beta^2}, \frac{1}{\sigma}\right) \tag{IV.26}$$

(F) Derivation of the Result of Eq. (73)

Using the result given in Eq. (IV.7), we may write the integral in Eq. (71) as follows:

$$\Delta L_1 = -\frac{\rho\Gamma_o^2}{4}\int_0^{\pi/2} \csc^2\theta \, d\theta \left\{ 1 - \frac{4}{\pi}\int_0^{\pi/2} \frac{\cos^2\phi \, d\phi}{\sqrt{1+\frac{1}{\lambda^2}\sin^2\theta \sin^2\phi}} \right\}$$

$$= -\frac{\rho\Gamma_o^2}{4}\left\{ -\cot\theta \left[1 - \frac{4}{\pi}\int_0^{\pi/2} \frac{\cos^2\phi \, d\phi}{\sqrt{1+\frac{1}{\lambda^2}\sin^2\theta \sin^2\phi}} \right]_{\theta\to 0}^{\pi/2} + \frac{4}{\pi\lambda^2}\int_0^{\pi/2} \cos^2\theta \, d\theta \int_0^{\pi/2} \frac{\sin^2\phi \cos^2\phi \, d\phi}{\left[1+\frac{1}{\lambda^2}\sin^2\theta \sin^2\phi\right]^{3/2}} \right\}$$

$$= -\frac{\rho\Gamma_o^2}{\pi\lambda^2}\int_0^{\pi/2} \cos^2\theta \, d\theta \int_0^{\pi/2} \frac{\sin^2\phi \cos^2\phi \, d\phi}{\left[1+\frac{1}{\lambda^2}\sin^2\theta - \frac{1}{\lambda^2}\sin^2\theta \sin^2\phi\right]^{3/2}}$$

$$= -\frac{\rho\Gamma_o^2}{\pi\lambda^2}\int_0^{\pi/2} \frac{\cos^2\theta}{\left[1+\frac{1}{\lambda^2}\sin^2\theta\right]^{3/2}} \, \acute{\mathcal{C}}\left(\frac{\sin\theta}{\sqrt{\lambda^2+\sin^2\theta}}\right) d\theta \qquad \text{(IV. 26)}$$

where $\acute{\mathcal{C}}(k)$ is a derived complete elliptic integral (cf. Ref. 25, p. 73) defined by

$$\acute{\mathcal{C}}(k) = \int_0^{\pi/2} \frac{\sin^2\phi \cos^2\phi}{\left[1 - k^2 \sin^2\phi\right]^{3/2}} \, d\phi \qquad \text{(IV. 27)}$$

Finally, changing the variable of integration from θ to k by

$$k = \frac{\sin\theta}{\sqrt{\lambda^2+\sin^2\theta}} \qquad \text{(IV. 27a)}$$

we obtain

$$\Delta L_1 = -\frac{\rho\Gamma_o^2}{\pi\lambda}\int_0^{\mathcal{J}} \sqrt{\frac{1-\left(\frac{k}{\mathcal{J}}\right)^2}{1-k^2}} \, \acute{\mathcal{C}}(k) \, dk \qquad \text{(IV. 28)}$$

where

$$\mathcal{J} = \frac{1}{\sqrt{1+\lambda^2}} \qquad \text{(IV. 28a)}$$

This is the form used in Eq. (73).

(G) Derivation of the Result of Eq. (76)

Substituting the expansion of $\acute{C}(k)$ given by Eq. (75) into Eq. (IV. 28), then applying the transformation $k^2 = \chi^2 t$ and integrating termwise, we have

$$\Delta L_1 = -\frac{\rho \Gamma_o^2}{32\lambda}\chi \int_0^1 (1-t)^{\frac{1}{2}}(1-\chi^2 t)^{-\frac{1}{2}} \left[1+6\frac{\chi^2}{8}t + \frac{75}{2}\left(\frac{\chi^2}{8}\right)^2 t^2 + 245\left(\frac{\chi^2}{8}\right)^3 t^3 + 0(\chi^8)\right]\frac{dt}{\sqrt{t}}$$

$$= -\frac{\pi\rho\Gamma_o^2}{64\lambda}\chi \left\{ {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2},2;\chi^2\right) + \frac{3}{2}\left(\frac{\chi^2}{8}\right) {}_2F_1\left(\tfrac{1}{2},\tfrac{3}{2};3;\chi^2\right)\right.$$

$$\left. + \frac{75}{16}\left(\frac{\chi^2}{8}\right)^2 {}_2F_1\left(\tfrac{1}{2},\tfrac{5}{2};4;\chi^2\right) + \frac{1225}{64}\left(\frac{\chi^2}{8}\right)^3 {}_2F_1\left(\tfrac{1}{2},\tfrac{7}{2};5;\chi^2\right) + 0\left[\left(\frac{\chi^2}{8}\right)^4\right]\right\}$$

(IV. 29)

where ${}_2F_1(a,b,c;z)$ denotes the hypergeometric function which has the following integral representation:

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \quad (R\ell c > R\ell b > 0).$$

Expanding these hypergeometric functions into convergent series, we finally have

$$\Delta L_1 = -\frac{\pi\rho\Gamma_o^2}{64\lambda\sqrt{1+\lambda^2}}\left\{1+\frac{5}{16}\frac{1}{(1+\lambda^2)} + \frac{181}{512}\frac{1}{(1+\lambda^2)^2} + \frac{3015}{8^5}\frac{1}{(1+\lambda^2)^3} + 0\left[\frac{1}{(1+\lambda^2)^4}\right]\right\}.$$

(IV. 30)

(H) Evaluation of the Integral in Eq. (79)

Applying the transformation $\sec\theta = \cosh\frac{u}{2}$ to the inner integral in Eq. (79), and proceeding in a similar way as shown in (IV. D), we have

$$\mathcal{J}_2(u) = \int_0^{\pi/2} e^{-\frac{2u}{\sigma}\sec^2\theta} \left[\frac{J_1(\frac{u}{\beta}\sec^2\theta\,\sin\theta)}{\sec^2\theta\,\sin\theta}\right]^2 \sec^4\theta\, d\theta$$

$$= \frac{1}{2}\, e^{-\frac{u}{\sigma}} \int_0^{\infty} e^{-\frac{u}{\sigma}\cosh u} \left[\frac{J_1(\frac{u}{2\beta}\sinh u)}{\frac{1}{2}\sinh u}\right]^2 \left(\frac{1+\cosh u}{2}\right) \cosh\frac{u}{2}\, du$$

$$= \frac{1}{\sqrt{\pi}}\, e^{-\frac{u}{\sigma}} \sum_{n=0}^{\infty} \frac{(-)^n\,\Gamma(n+\frac{3}{2})}{n!\,(n+1)!\,(n+2)!} \left(\frac{u}{2\beta}\right)^{2n+2} \int_0^{\infty} e^{-\frac{u}{\sigma}\cosh u} \sinh^{2n}u\,(1+\cosh u)\cosh\frac{u}{2}\, du$$

Now $\sinh^{2n}u$ can be expanded in terms of $\cosh m u$ as follows

$$\sinh^{2n}u = \frac{(e^{u}-e^{-u})^{2n}}{2^{2n}} = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{m=0}^{n-1}(-)^m\binom{2n}{m}\cosh(2n-2m)u\,,$$

and

$$(1+\cosh u)\cosh\frac{u}{2} = \frac{1}{2}\cosh\frac{3u}{2} + \frac{3}{2}\cosh\frac{u}{2}\,.$$

Using the relation (cf. Ref. 24, p. 181)

$$\int_0^{\infty} e^{-z\cosh u}\cosh \nu u\, du = K_\nu(z), \qquad (R\ell\ \nu > 0,\ R\ell\ z > 0)\,, \qquad \text{(IV. 31)}$$

we then have

$$\mathcal{J}_2(u) = \frac{2}{\sqrt{\pi}}\, e^{-\frac{u}{\sigma}} \sum_{n=0}^{\infty} \frac{(-)^n\,\Gamma(n+\frac{3}{2})}{n!\,(n+1)!\,(n+2)!}\left(\frac{u}{4\beta}\right)^{2n+2} \left\{ \binom{2n}{n}\left[K_{\frac{3}{2}}\left(\frac{u}{\sigma}\right) + 3K_{\frac{1}{2}}\left(\frac{u}{\sigma}\right)\right]\right.$$

$$+ \sum_{m=0}^{n-1}(-)^m\binom{2n}{m}\left[K_{2(n-m)+\frac{3}{2}}\left(\frac{u}{\sigma}\right) + 3K_{2(n-m)+\frac{1}{2}}\left(\frac{u}{\sigma}\right) + 3K_{2(n-m)-\frac{1}{2}}\left(\frac{u}{\sigma}\right)\right.$$

$$\left.\left. + K_{2(n-m)-\frac{3}{2}}\left(\frac{u}{\sigma}\right)\right]\right\}. \qquad \text{(IV. 32)}$$

When σ is very large, then (cf. Ref. 24, p. 80)

$$K_{n+\frac{1}{2}}\left(\frac{u}{\sigma}\right) = \sqrt{\frac{\pi\sigma}{2u}}\, e^{-\frac{u}{\sigma}} \frac{(2n)!}{n!\,(2\frac{u}{\sigma})^n}\left(1+0(\tfrac{1}{\sigma})\right) = \sqrt{\frac{u}{2\sigma}}\, e^{-\frac{u}{\sigma}} \frac{2^n\,\Gamma(n+\frac{1}{2})}{(\frac{u}{\sigma})^{n+1}}\left(1+0(\tfrac{1}{\sigma})\right). \qquad \text{(IV. 33)}$$

It can be seen that the first order term of $\mathcal{J}_2(u)$ comes from the term with $K_{2n+\frac{3}{2}}(\frac{u}{\sigma})$, hence for large σ, $\mathcal{J}_2(u)$ may be approximated as

$$\mathcal{J}_2(u) \doteq \sqrt{\frac{u}{2\pi\sigma}}\, e^{-\frac{2u}{\sigma}} \left[\sum_{n=0}^{\infty} \frac{(-)^n \Gamma(n+\frac{3}{2})\Gamma(2n+\frac{3}{2})}{n!(n+1)!(n+2)!} \left(\frac{\sigma}{2\beta}\right)^{2n+2} \right] (1 + 0(\tfrac{1}{\sigma}))$$

$$\doteq \frac{1}{4\pi} \sqrt{\frac{u}{\sigma}}\, e^{-\frac{2u}{\sigma}} \left(\sqrt{\pi}\,\Gamma(\tfrac{1}{4})\right) \left[1 - \frac{\lambda}{\sqrt{1+\lambda^2}} F(\tfrac{1}{2}, \tfrac{3}{4}; 2; \tfrac{1}{1+\lambda^2}) \right] (1 + 0(\tfrac{1}{\sigma}))$$

$$\doteq \frac{\Gamma(\frac{1}{4})}{4\sqrt{\pi}} \sqrt{\frac{u}{\sigma}}\, e^{-\frac{2u}{\sigma}} \left[1 - \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}\,\Gamma(\frac{1}{4})} \frac{\lambda}{\sqrt{1+\lambda^2}} \right] (1 + 0(\lambda^2, \tfrac{1}{\sigma})), \quad \sigma \to \infty$$

(IV. 34)

For small values of σ (h, λ, large), it can be shown by the method of steepest descent that $\mathcal{J}_2$ has the same asymptotic value as $\mathcal{J}_1$ given in Eq. (66), that is

$$\mathcal{J}_2(u) \doteq \frac{1}{8}\, e^{-\frac{2u}{\sigma}} \sqrt{\frac{\pi\sigma}{2}}\, \frac{u^{\frac{3}{2}}}{\beta^2} (1 + 0(\sigma)) , \quad \text{as} \quad \sigma \to 0 . \qquad \text{(IV. 35)}$$

Therefore, for σ large, we have from Eqs. (79) and (IV. 34)

$$\Delta L_2 = -\frac{\rho\Gamma_0}{4\sqrt{\pi}} \Gamma(\tfrac{1}{4}) \frac{1}{\sqrt{\sigma}} \left(1 - \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}\,\Gamma(\frac{1}{4})} \frac{\lambda}{\sqrt{1+\lambda^2}}\right) \int_0^{\infty} \frac{e^{-\frac{2u}{\sigma}}}{\sqrt{u(u-1)}} du \; (1 + 0(\lambda^2, \tfrac{1}{\sigma}))$$

$$= -\frac{\rho\Gamma_0}{4\sqrt{\pi}} \Gamma(\tfrac{1}{4}) \frac{1}{\sqrt{\sigma}} \left(1 - \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}\,\Gamma(\frac{1}{4})} \frac{\lambda}{\sqrt{1+\lambda^2}}\right) \left[-\sqrt{\frac{\pi\sigma}{2}} + e^{-\frac{2}{\sigma}} \int_{-1}^{\infty} \frac{\sqrt{t+1}}{t} e^{-\frac{2}{\sigma}t} dt \right]$$

$$(1 + 0(\lambda^2, \tfrac{1}{\sigma}))$$

$$\doteq +\frac{\rho\Gamma_0}{4\sqrt{\pi}} \Gamma(\tfrac{1}{4}) \frac{1}{\sqrt{\sigma}} \left(1 - \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}\,\Gamma(\frac{1}{4})} \frac{\lambda}{\sqrt{1+\lambda^2}}\right) \left[\sqrt{\frac{\pi\sigma}{2}} + e^{-\frac{2}{\sigma}} E_i(\tfrac{2}{\sigma}) \right] (1 + 0(\lambda^2, \tfrac{1}{\sigma})) ,$$

where $E_i(t)$ represents the exponential integral function defined by

$$E_i(t) = -\int_{-t}^{\infty} \frac{e^{-x} dx}{x} \qquad \text{(t real, cf. Ref. 23, p. 471)}$$

which has the following expansion

$$E_i(\frac{2}{\sigma}) = \gamma + \log\frac{2}{\sigma} + \frac{2}{\sigma} + \frac{1}{2!\,2!}(\frac{2}{\sigma})^2 + \frac{1}{3!\,3!}(\frac{2}{\sigma})^3 + \ldots\ldots, \text{ for } \sigma \text{ large.} \tag{IV.36}$$

Finally we have

$$\Delta L_2 = \frac{\rho\Gamma_o}{4\sqrt{2}}\Gamma(\frac{1}{4})\left(1 - \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}\,\Gamma(\frac{1}{4})}\frac{\lambda}{\sqrt{1+\lambda^2}}\right)\left[1 + \sqrt{\frac{2}{\pi\sigma}}(\gamma + \log\frac{2}{\sigma})\right](1 + 0(\lambda^2, \frac{1}{\sigma})), \text{ as } \sigma \to \infty \tag{IV.37}$$

On the other hand, for σ small, the asymptotic value of ΔL_2 is

$$\Delta L_2 \cong -\frac{\rho\Gamma_o^2}{8}\sqrt{\frac{\pi\sigma}{2}}\frac{1}{\beta^2}\int_o^{\infty} e^{-\frac{2u}{\sigma}}\frac{\sqrt{u}}{u-1}du\ (1 + 0(\sigma))$$

$$\cong -\frac{\rho\Gamma_o^2}{8}\sqrt{\frac{\pi\sigma}{2}}\frac{1}{\beta^2}e^{-\frac{2}{\sigma}}\int_{-\frac{2}{\sigma}}^{\infty} e^{-t}\frac{(1+\frac{\sigma}{2}t)^{1/2}}{t}dt \quad (1 + 0(\sigma))$$

$$\cong \frac{\rho\Gamma_o^2}{8}\sqrt{\frac{\pi\sigma}{2}}\frac{1}{\beta^2}e^{-\frac{2}{\sigma}}E_i(\frac{2}{\sigma})\ (1 + 0(\sigma))\,.$$

The asymptotic expansion of $E_i(\frac{2}{\sigma})$ for $\frac{2}{\sigma}$ large is

$$E_i(\frac{2}{\sigma}) \cong e^{\frac{2}{\sigma}}(\frac{\sigma}{2})\ (1 + \frac{\sigma}{2} + 2!\,(\frac{\sigma}{2})^2 + 0(\sigma^3))\,.$$

Hence,

$$\Delta L_2 \cong \frac{\rho\Gamma_o^2}{8}\sqrt{\pi}\,(\frac{\sigma}{2})^{\frac{3}{2}}\frac{1}{\beta^2}(1 + \frac{\sigma}{2} + 0(\sigma^2)), \quad \text{as } \sigma \to 0. \tag{IV.38}$$

(I) Derivation of the Result of Eq. (83)

From Eq. (82a), we have

$$u_1(y) = \left(\frac{\partial\varphi_2}{\partial x}\right)_{x=0} = -\frac{\Gamma_o}{2\pi b}\int_0^{\pi/2} d\theta\int_0^{\infty} e^{-2\lambda\mu}\,\frac{J_1(\mu\sin\theta)}{\sin\theta}\cos(\mu\eta\sin\theta)\,d\mu$$

$$= -\frac{\Gamma_o}{4\pi b}\int_0^{\pi/2} d\theta\int_0^{\infty} e^{-2\lambda\mu}\left(\sum_{n=0}^{\infty}\frac{(-)^n\left(\frac{\mu\sin\theta}{2}\right)^{2n}}{n!\,(n+1)!}\right)\left(\sum_{m=0}^{\infty}\frac{(-)^m(\mu\eta\sin\theta)^{2m}}{(2m)!}\right)\mu\,d\mu$$

$$= -\frac{\Gamma_o}{4\pi b}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-)^{n+m}\,\sqrt{\pi}\,\eta^{2m}}{n!\,(n+1)!\,2^{2(n+m)}\,\Gamma(m+1)\Gamma(m+\frac{1}{2})}\left(\int_0^{\infty} e^{-2\lambda\mu}\,\mu^{2(n+m)+1}d\mu\right)\left(\int_0^{\pi/2}\sin^{2(m+n)}\theta\,d\theta\right)$$

$$= -\frac{\Gamma_o}{4\pi b}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-)^{n+m}\,\sqrt{\pi}\,\eta^{2m}}{n!\,(n+1)!\,2^{2(n+m)}\,m!\,\Gamma(m+\frac{1}{2})}\;\frac{\Gamma(n+m+\frac{1}{2})\Gamma(n+m+\frac{3}{2})}{4\lambda^{2n+2m+2}}$$

$$= -\frac{\Gamma_o}{4\sqrt{\pi}\,b}\;\frac{1}{4\lambda^2}\sum_{m=0}^{\infty}\frac{(-)^m\left(\frac{\eta}{2\lambda}\right)^{2m}\Gamma(m+\frac{3}{2})}{m!}\;F\left(m+\frac{1}{2},\;m+\frac{3}{2};2;-\frac{1}{4\lambda^2}\right)\;.$$

Now if we continue the hypergeometric function analytically to region $0\leqslant\lambda<\infty$, we obtain

$$u_1(y) = -\frac{\Gamma_o}{4\sqrt{\pi}\,b}\;\frac{1}{2\lambda\sqrt{1+4\lambda^2}}\sum_{m=0}^{\infty}\frac{(-)^m(\eta)^{2n}\Gamma(m+\frac{3}{2})}{m!}\left(\frac{1}{1+4\lambda^2}\right)^m F\left(m+\frac{1}{2},\frac{1}{2}-m;2;\frac{1}{1+4\lambda^2}\right) \tag{IV. 39}$$

(J) Derivation of the Result of Eq. (85)

Applying the transformation $\mu = (b\kappa\sec^2\theta)u$ to Eq. (82b), we have

$$u_2(y) = -\frac{\kappa\Gamma_o}{\pi}\int_0^{\infty}\frac{du}{u-1}\int_0^{\pi/2} e^{-\frac{2u}{\sigma}\sec^2\theta}\,\frac{J_1(\frac{u}{\beta}\sec^2\theta\sin\theta)}{\sec^2\sin\theta}\cos\left(\frac{u\eta}{\beta}\sec^2\theta\sin\theta\right)\sec^4\theta\,d\theta \tag{IV. 40}$$

where

$$\sigma = \frac{1}{h\kappa}\;,\qquad \beta = \frac{1}{b\kappa}\;,\qquad \eta = \frac{y}{b}\;.$$

The inner integral can be treated in a way similar to that applied to $\mathcal{J}_2$ in (IV.H). In particular, at $y = 0$, we have

$$u_2(0) = -\frac{\kappa\Gamma_0}{4\pi}\int_0^\infty e^{-\frac{u}{\sigma}}\frac{du}{u-1}\int_0^\infty e^{-\frac{u}{\sigma}\cosh t}\left[\frac{J_1(\frac{u}{2\beta}\sinh t)}{\frac{1}{2}\sinh t}\right](1+\cosh t)\cosh\frac{t}{2}\,dt$$

$$= -\frac{\kappa\Gamma_0}{8\pi\beta}\int_0^\infty e^{-\frac{u}{\sigma}}\frac{u}{u-1}\,du\sum_{n=0}^{\infty}\frac{(-)^n(\frac{u}{4\beta})^{2n}}{n!\,(n+1)!}\int_0^\infty e^{-\frac{u}{\sigma}\cosh t}\sinh^{2n}t\,(1+\cosh t)\cosh\frac{t}{2}\,dt$$

$$= -\frac{\kappa\Gamma_0}{16\pi\beta}\int_0^\infty e^{-\frac{u}{\sigma}}\frac{u}{u-1}\,du\sum_{n=0}^{\infty}\frac{(-)^n(\frac{u}{8\beta})^{2n}}{n!\,(n+1)!}\left\{\binom{2n}{n}\left[K_{\frac{3}{2}}(\tfrac{u}{\sigma})+3K_{\frac{1}{2}}(\tfrac{u}{\sigma})\right]\right.$$

$$+\sum_{m=0}^{n-1}(-)^m\binom{2n}{m}\left[K_{2(n-m)+\frac{3}{2}}(\tfrac{u}{\sigma})+3K_{2(n-m)+\frac{1}{2}}(\tfrac{u}{\sigma})+3K_{2(n-m)-\frac{1}{2}}(\tfrac{u}{\sigma})\right.$$

$$\left.\left.+K_{2(n-m)-\frac{3}{2}}(\tfrac{u}{\sigma})\right]\right\}.$$

For σ very large, the first order term of $u_2(0)$ comes from

$$K_{2n+\frac{3}{2}}(\tfrac{u}{\sigma}) \cong \sqrt{\frac{u}{2\sigma}}\,e^{-\frac{u}{\sigma}}\frac{2^{2n+1}\Gamma(2n+\frac{3}{2})}{(\frac{u}{\sigma})^{2n+2}}\,(1+O(\tfrac{1}{\sigma}))$$

$$\cong \frac{2}{\sqrt{\pi}}\left(\frac{\sigma}{u}\right)^{3/2}e^{-\frac{u}{\sigma}}\frac{4^{2n}}{(\frac{u}{\sigma})^{2n}}\Gamma(n+\tfrac{3}{4})\Gamma(n+\tfrac{5}{4})\,(1+O(\tfrac{1}{\sigma}))\,.$$

Hence,

$$u_2(0) \cong -\frac{(\kappa\Gamma_0)\sigma^{3/2}}{4\pi\sqrt{\pi}\,\beta}\int_0^\infty e^{-\frac{2u}{\sigma}}\frac{du}{\sqrt{u}\,(u-1)}\sum_{n=0}^{\infty}\frac{(-)^n\Gamma(n+\frac{3}{4})\Gamma(n+\frac{5}{4})}{n!\,(n+1)!}\left(\frac{1}{4\lambda^2}\right)^n(1+O(\tfrac{1}{\sigma}))$$

$$\cong -\frac{(\kappa\Gamma_0)\sigma^{3/2}}{4\pi\sqrt{\pi}\,\beta}\,\frac{\Gamma(\frac{3}{4})\Gamma(\frac{5}{4})}{\Gamma(2)}\,F(\tfrac{3}{4},\tfrac{5}{4};2;-\tfrac{1}{4\lambda^2})\int_0^\infty e^{-\frac{2u}{\sigma}}\frac{du}{\sqrt{u}\,(u-1)}\,(1+O(\tfrac{1}{\sigma}))\,.$$

Since

$$\Gamma(\tfrac{3}{4})\Gamma(\tfrac{5}{4}) = \frac{\pi}{4\sin\frac{3\pi}{4}} = \frac{\sqrt{2}\,\pi}{4},$$

$$F(\tfrac{3}{4},\tfrac{5}{4};2;-\frac{1}{4\lambda^2}) = \frac{(2\lambda)^{3/2}}{(1+4\lambda^2)^{3/4}} F(\tfrac{3}{4},\tfrac{3}{4};2;\frac{1}{1+4\lambda^2})$$

and (cf. Eq. IV.37)

$$\int_0^\infty e^{-\frac{2u}{\sigma}} \frac{du}{\sqrt{u(u-1)}} \cong \sqrt{\frac{\pi\sigma}{2}}\left[1+\sqrt{\frac{2}{\pi\sigma}}\,(\gamma+\log\frac{2}{\sigma})\right]\ (1+0(\frac{1}{\sigma}))\,,$$

we finally obtain

$$u_2(0) \cong -\frac{\Gamma_0}{4b}\frac{1}{\sqrt{2\lambda}}\frac{1}{(1+4\lambda^2)^{3/4}} F(\tfrac{3}{4},\tfrac{3}{4};2;\frac{1}{1+4\lambda^2})\left[1+\sqrt{\frac{2}{\pi\sigma}}(\gamma+\log\frac{2}{\sigma})\right](1+0(\frac{1}{\sigma}))\,. \tag{IV.41}$$

This formula is good for all λ, $0 \leqslant \lambda < \infty$, provided that σ is large. The above hypergeometric function can be expressed in terms of tabulated elliptic integrals by using Kummer's transformation and Gauss' recursion formula as follows:

$$F(\tfrac{3}{4};\tfrac{3}{4};2;\frac{1}{1+4\lambda^2}) = F(\tfrac{3}{2},\tfrac{3}{2};2;\tfrac{1}{2}-\frac{\lambda}{\sqrt{1+4\lambda^2}}) \qquad \text{(Kummer)}$$

$$= \frac{2}{k^2}\left[F(\tfrac{1}{2},\tfrac{3}{2};1;k^2) - F(\tfrac{1}{2},\tfrac{1}{2};1;k^2)\right] \qquad \text{(Gauss)}$$

where

$$k^2 = \frac{1}{2} - \frac{\lambda}{\sqrt{1+4\lambda^2}}\,.$$

Applying the transformation formula once more, we obtain

$$F(\tfrac{3}{4},\tfrac{3}{4};2;\frac{1}{1+4\lambda^2}) = \frac{2}{k^2}\left(\frac{1}{1-k^2}F(\tfrac{1}{2},-\tfrac{1}{2};1;k^2) - F(\tfrac{1}{2},\tfrac{1}{2};1;k^2)\right)$$

$$= \frac{4}{\pi}\frac{1}{k^2(1-k^2)}\left[E(k)-(1-k^2)K(k)\right] = \frac{4}{\pi}\frac{1}{1-k^2}B(k)$$

$$= \frac{4}{\pi}\frac{2\sqrt{1+4\lambda^2}}{\sqrt{1+4\lambda^2}+2\lambda} B\left(\sqrt{\frac{1}{2}-\frac{\lambda}{\sqrt{1+4\lambda^2}}}\right)\,.$$

where $B(k)$ is defined by Eq. (7 a). Substituting this relation into (IV.40), we have

$$u_2(0) \cong -\frac{2\Gamma_o}{\pi b}\frac{1}{\sqrt{2\lambda}}\frac{1}{(1+4\lambda^2)^{1/4}}\frac{1}{\sqrt{1+4\lambda^2}+2\lambda}\; B\left(\sqrt{\frac{1}{2}-\frac{\lambda}{\sqrt{1+4\lambda^2}}}\right)(1+0(\tfrac{1}{\sigma}))\,. \tag{IV.42}$$

When σ is very small, we can apply the method of steepest descent to the inner integral of Eq. (IV.40). The result is as follows:

$$u_2(y) \cong -\frac{\kappa\Gamma_o}{\pi}\frac{\sqrt{2\pi\sigma}}{8\beta}\int_o^\infty e^{-\frac{2u}{\sigma}}\frac{\sqrt{u}}{u-1}\,du\;(1+0(\sigma))$$

$$\cong +\frac{\kappa\Gamma_o}{8\beta}\sqrt{\frac{2\sigma}{\pi}}\,(\tfrac{\sigma}{2})\;(1+0(\sigma)) \qquad \text{(cf. Section H, Eq. IV.38)}$$

$$\cong \frac{\Gamma_o}{8b}\frac{1}{\sqrt{2\pi\sigma}}\frac{1}{\lambda^2}\;(1+0(\sigma)) \tag{IV.43}$$

(K) Derivation of the Result of Eq. (91)

The method used to approximate

$$\varphi_{4_z}(0,0,-h) = -\kappa\Gamma_o\int_o^{\pi/2} e^{-\frac{2}{\sigma}\sec^2\theta}\;\frac{J_1(\frac{1}{\beta}\sec^2\theta\sin\theta)}{\sec^2\theta\sin\theta}\;\sec^5\theta\,d\theta$$

for σ large is very similar to that discussed in (IV.D and E). Introducing the transformation $\sec\theta = \cosh\frac{u}{2}$ to the above integral and then expand J_1 into a series, integrating termwise, we obtain

$$\varphi_{4_z}(0,0,-h) = -\frac{\kappa\Gamma_o}{2}e^{-\frac{1}{\sigma}}\int_o^\infty e^{-\frac{1}{\sigma}\cosh u}\;\frac{J_1(\frac{1}{2\beta}\sinh u)}{\frac{1}{2}\sinh u}\;\left(\frac{1+\cosh u}{2}\right)^2 du$$

$$= -\frac{\kappa\Gamma_o}{16\beta}e^{-\frac{1}{\sigma}}\sum_{n=o}^\infty\frac{(-)^n(\frac{1}{4\beta})^{2n}}{n!\,(n+1)!}\int_o^\infty e^{-\frac{1}{\sigma}\cosh u}\sinh^{2n}u\,(1+\cosh u)^2\,du$$

$$= -\frac{\kappa\Gamma_o}{8\sqrt{\pi}\,\beta}e^{-\frac{1}{\sigma}}\sum_{n=o}^\infty\frac{(-)^n\Gamma(n+\frac{1}{2})}{n!\,(n+1)!}\left(\frac{\sigma}{8\beta^2}\right)^n\left[K_n(\tfrac{1}{\sigma})+(1+\sigma(n+\tfrac{1}{2}))K_{n+1}(\tfrac{1}{\sigma})\right] \tag{IV.44}$$

where the relations given by Eq. (IV.9) have been used. Referring to the expression of $K_n(\frac{1}{\sigma})$ for σ large given by Eq. (IV.11), it is easy to see that the first order term of φ_{4_z} comes from the term $\frac{1}{2}(n!)(2\sigma)^{n+1}$ of the expansion of $K_{n+1}(\frac{1}{\sigma})$. Then

$$\varphi_{4_z}(0,0,h) \cong - \frac{\kappa \Gamma_o}{8\sqrt{\pi}\beta} e^{-\frac{1}{\sigma}} \sum_{n=o}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2})}{n!\,(n+1)!} \left(\frac{\sigma}{8\beta^2}\right)^n \left[\sigma(n+\frac{1}{2}) \frac{n!}{2}(2\sigma)^{n+1} \left(1+O(\frac{1}{\sigma}, \frac{1}{\beta})\right)\right],$$

$$\cong - \frac{\kappa \Gamma_o}{2\sqrt{\pi}} \beta \sum_{n=o}^{\infty} \frac{(-)^n \Gamma(n+\frac{3}{2})}{(n+1)!} \left(\frac{1}{4\lambda^2}\right)^{n+1} \left[1+O(\frac{1}{\sigma}, \frac{1}{\beta})\right]$$

$$\cong - \frac{\kappa \Gamma_o \beta}{2} \left\{1 - \sum_{n=o}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\, n!} \left(-\frac{1}{4\lambda^2}\right)^n \right\} \left[1+O(\frac{1}{\sigma}, \frac{1}{\beta})\right]$$

$$\cong - \frac{\Gamma_o}{2b} \left\{1 - \left(1+\frac{1}{4\lambda^2}\right)^{-\frac{1}{2}} \right\} \left[1+O(\frac{1}{\sigma}, \frac{1}{\beta})\right]$$

$$\cong - \frac{\Gamma_o}{2b} \left[1 - \frac{2\lambda}{\sqrt{1+4\lambda^2}}\right] \left[1+O(\frac{1}{\sigma}, \frac{1}{\beta})\right] \qquad \text{(IV.45)}$$

where the appropriate analytic continuation from $\lambda > \frac{1}{2}$ to $0 \leqslant \lambda < \infty$ is made at the last step.

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